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Holographic QCD: Past, Present, and Future

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Abstract

At the dawn of a new theoretical tool based on the AdS/CFT correspondence for nonperturbative aspects of quantum chromodynamics, we give an interim review on the new tool, holographic QCD, with some of its accomplishment. We try to give an A-to-Z picture of the holographic QCD, from string theory to a few selected top-down holographic QCD models with one or two physical applications in each model. We may not attempt to collect diverse results from various holographic QCD model studies.

Contents

1 Introduction

Quantum chromodynamics (QCD) is widely believed to be the fundamental theory of the strong interaction. Despite its tremendous success in high energy hadronic phenomenology based on perturbation theory, QCD is not directly applicable to every physical system governed by the strong interaction.

Especially, at low energies due to the largeness of the coupling constant involved, the standard perturbation method is not viable in most cases. There have been many interesting and (partly if not fully) qualified approaches for the strong interaction at low energy. In this article, we review one of them which is flourishing these days in describing QCD (or its cousins) in a nonperturbative regime.

Based on the AdS/CFT correspondence (or the gauge/gravity duality, its generalized version) [1–3], interesting and important attempts have been made to understand nonperturbative aspects of QCD under the name "holographic QCD", see [4–11] for some prototypical models. There are in general two different routes to arrive at the holographic dual description of QCD. One way is, so-called, a top-down approach based on stringy D-brane configurations. The other way is a bottom-up approach, in which a 5D holographic QCD model is constructed from QCD.

In this interim review, for those who are not experts in holographic QCD, we review holographic QCD models starting from a concise but essential summary of string theory including the AdS/CFT correspondence. Then, we will introduce some frequently used top-down models and present a few selected sample works for QCD (or more precisely QCD-like) phenomena in each model. We will not attempt to show innumerable outcomes out of holographic QCD models, though some of them are quite interesting and influential, for instance, the universal viscosity to entropy ratio [12]. For more on holographic QCD, we may refer to [13–16] and [17] for a summary focused on results from bottom-up models.

After the introduction of basic string theory and the AdS/CFT correspondence, we will start with the D3/D7 model. In this D3/D7 model, we will show a few results on the probe of the quark-gluon plasma (QGP). We will then describe D4/D8 model with a touch on holographic baryons and nuclear forces. Lastly, we make a brief summary of the D4/D6 model and discuss nuclear symmetry energy and nuclear to strange matter transition. To demonstrate the usefulness and versatility of the holographic QCD approach and its recent contributions to QCD (or QCD-like) phenomenology, we briefly mention some of recent results from various holographic studies in section 8. Some remarks will close this write-up.

2 Basics of string theory

We start by giving some basic aspects about general relativity and string theory. In this chapter, we shall introduce main ingredients and set up the stage for the next sections. We start with recalling the Einstein's general relativity. The standard text books can be found in [18] for string theory and in [19] for gravity theory.

2.1 Einstein gravity

The idea of Einstein gravity is that the spacetime geometry is dynamical. We work in D-dimensional spacetime with metric $G_{MN}(x)$,

$$ds^2 = G_{MN}\mathrm{d}x^M\mathrm{d}x^N, \tag{2.1}$$

whose signature is $(- + \cdots +)$. The dynamics of spacetime can be determined by

$$S_{\mathrm{EH}} = \frac{1}{16\pi G_{\mathrm{N}}^{(D)}}\int_{\mathcal{M}}\mathrm{d}^Dx\sqrt{-G}R = \frac{1}{2\kappa^2}\int_{\mathcal{M}}\mathrm{d}^Dx\sqrt{-G}R, \tag{2.2}$$

where $G = \det G_{MN}$ is the determinant of the metric, and $R(x)$ is the scalar curvature. The action (2.2) is called as the Einstein-Hilbert action which contains second derivative terms of the metric $G_{MN}(x)$. $G_{\mathrm{N}}^{(D)}(= \kappa^2/(8\pi))$ is a D-dimensional Newton constant and its mass dimension is mass$^{(2-D)}$ which has a negative power for $D > 2$. Depending on the spacetime topologies and setups to be

considered, one could add several possible diffeomorphism invariant terms such as the boundary terms, the cosmological constant, and more higher derivative contributions to the Einstein-Hilbert action. We define D-dimensional Planck length $l_{\rm P}^{(D)}$ as $l_{\rm P}^{(D)} = (8\pi G_{\rm N}^{(D)})^{1/(D-2)}$ in the dimensional analysis of the action (2.2).

The couplings to matter fields $\psi(x)$ are given through the metric $G_{MN}(x)$ and the Christoffel symbol ${\Gamma^M}_{KL}(x)$[1] which is defined by the metric. Introducing a matter action as $S_{\rm matter}[\psi, G_{MN}]$, the variation of the total action $S_{\rm EH} + S_{\rm matter}$ with respect to the metric $G^{MN}(x)$ gives the Einstein equation

$$R_{MN} - \frac{1}{2}G_{MN}R = 8\pi G_{\rm N}^{(D)} T_{MN}, \tag{2.3}$$

where $R_{MN}(x)$ is the Ricci tensor and $T_{MN}(x)$ is the energy momentum tensor defined by

$$T_{MN} \equiv -\frac{2}{\sqrt{-G}}\frac{\delta S_{\rm matter}}{\delta G^{MN}}. \tag{2.4}$$

Let us consider the perturbation around the flat Minkowski spacetime $\eta_{MN} = {\rm diag}(-,+,\cdots,+)$,

$$G_{MN}(x) = \eta_{MN} + \kappa \widehat{h}_{MN}(x) + \mathcal{O}(\widehat{h}^2), \tag{2.5}$$

where the coupling constant $\kappa = \sqrt{8\pi G_{\rm N}^{(D)}}$ has been introduced such that the kinetic term of the graviton $\widehat{h}_{MN}(x)$ is canonically normalized. Up to total derivatives, the action (2.2) can be written as

$$\begin{aligned} S_{\rm EH} = \int {\rm d}^D x \bigg\{ &\frac{1}{2}(\partial_M \widehat{h}^M{}_N)(\partial_L \widehat{h}^{LN}) - \frac{1}{2}(\partial_M \widehat{h}_L{}^L)(\partial_N \widehat{h}^{MN}) + \frac{1}{4}(\partial_M \widehat{h}_L{}^L)(\partial^M \widehat{h}_K{}^K) - \frac{1}{4}(\partial_M \widehat{h}_{NL})(\partial^M \widehat{h}^{NL}) \\ &+\kappa\Big(\widehat{h}(\partial\widehat{h})^2 + \cdots\Big) + \mathcal{O}(\kappa^2)\bigg\}, \end{aligned} \tag{2.6}$$

where the derivative coupling term, which is proportional to κ, has been symbolically displayed. By using the De Donder gauge $0 = \partial_N \widehat{h}_M{}^N(x) - \frac{1}{2}\partial_M \widehat{h}_N{}^N(x)$ to fix the general coordinate transformation $\delta \widehat{h}_{MN}(x) = \partial_M \varepsilon_N(x) + \partial_N \varepsilon_M(x)$, we can obtain the graviton propagator[2]

$$\langle \widehat{h}_{MN}(-k)\widehat{h}_{KL}(k)\rangle = -\frac{2i}{k^2}\Big(\eta_{MK}\eta_{NL} + \eta_{ML}\eta_{NK} - \frac{2}{D-2}\eta_{MN}\eta_{KL}\Big). \tag{2.7}$$

The action of the matter field is also expanded as

$$S_{\rm matter} = S_{\rm matter}[\psi, \eta_{MN}] - \int {\rm d}^D x \bigg\{ \frac{\kappa}{2} T_{MN}\widehat{h}^{MN} + \mathcal{O}(\kappa^2)\bigg\}. \tag{2.8}$$

Due to the coupling constant with a negative mass dimension, conventional perturbative methods do not work due to bad behavior at UV. Only in the low energies

$$\omega \ll \frac{1}{\kappa^{\frac{2}{D-2}}}, \tag{2.9}$$

[1]For fermions, we need to introduce the local Lorentz frame defined by vielbein, and the minimal coupling is given through the spin connections.

[2]As usual, we add the gauge fixing term to the action including the Faddeev-Popov ghost $\eta_M(x)$

$$\mathcal{L}_{\rm gauge-fixing} = -\frac{1}{2}\Big(\partial_N \widehat{h}_M{}^N - \frac{1}{2}\partial_M \widehat{h}_N{}^N\Big)^2 - \partial_M \bar{\eta}_N \partial^M \eta^N + \mathcal{O}(\widehat{h}^2).$$

the perturbation, in which the dimensionless coupling for given characteristic energy scale ω is $\omega\kappa^{2/(D-2)}$, makes a sense. In order to realize the low energy limit, it is sometimes convenient to send the energy scale to infinity or equivalently take a limit

$$\kappa \to 0 \tag{2.10}$$

with keeping the energies constant. In this low energy regime, the interactions turn to be weaker and under control.

A similar well-known example suffering from the UV divergence can be found in the electroweak theory. The electroweak theory was originally described by the four-Fermi interaction with a negative mass dimension coupling constant. This interaction is not renormalizable and breaks the perturbation theory at sufficient high energies. In this case, in high energy, we know new degrees of freedom come into the theory, i.e. W and Z bosons which mediate the contact interaction, and the UV divergence becomes soft. Indeed, the theory turns to be well-defined in the sense of the renormalization.

String theory is one of the formulations with well-defined consistent quantum gravity. Essential departure from the conventional field theory of point particle is to consider one-dimensional extended object i.e. string as a fundamental variable. Indeed, new (infinitely many) degrees of freedom just like W and Z bosons can be introduced. String world sheets which are formed by the string trajectories could restrict their amplitudes through their topologies. As a result, one can calculate consistent finite perturbative quantum gravity amplitudes.

In string theories, there exist not only strings but also various extended objects called p-brane which is a p-dimensional extended object. These extended objects move in the bulk D-dimensional spacetime and their trajectories become hypersurfaces embedded in the bulk. From the causality point of view, the hypersurface should be timelike, meaning that this should include a time direction. In this notation, for example, a point particle is 0-brane whose trajectory describes the world line and a string is 1-brane which sweeps out the world sheet in the bulk spacetime. The 2-brane is a "membrane" and the higher p-dimensional extended object is called as "p-brane". The free motion of the p-branes is described by the minimal action principle for the length, the surface, and the volumes. In the next subsections, we formulate these extended objects.

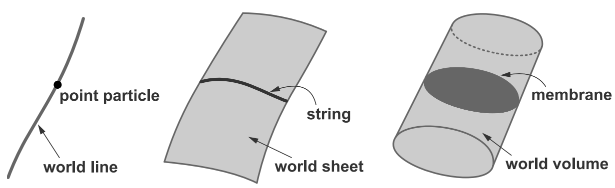

Figure 1: Extended objects

2.2 Point particle

The action of the particle with mass m can be given by the length of the "timelike" world line γ. In order to obtain the dynamical equation of motion, we need to introduce the target space coordinates $X^M(\lambda)$ with $M = 0, 1, \cdots, D-1$ which are fields on the world line parameterized by λ. The coordinates $X^M(\lambda)$ provide the location of the particle in the spacetime, i.e. the embedding coordinates of the world line into the bulk. Then, the action is given by

$$S_0 = -m \times (\text{length}) = -m \int_\gamma \mathrm{d}s = -m \int_\gamma \sqrt{-\mathrm{d}X^M \mathrm{d}X^N G_{MN}(X)} = -m \int_{\lambda_0}^{\lambda_1} \mathrm{d}\lambda \sqrt{-g_{\lambda\lambda}(\lambda)}, \tag{2.11}$$

where we have introduced

$$g_{\lambda\lambda}\big(X(\lambda)\big) = \frac{\mathrm{d}X^M}{\mathrm{d}\lambda}\frac{\mathrm{d}X^N}{\mathrm{d}\lambda} G_{MN}(X) \tag{2.12}$$

which is the "induced metric" on the world line. Apart from the manifest general covariance, the action is invariant under the change $\lambda \to \lambda'(\lambda)$ with the boundary conditions $\lambda'(\lambda_0) = \lambda_0$ and $\lambda'(\lambda_1) = \lambda_1$.

In general, there is no natural choice of the world line parameterization. One could fix this invariance in various ways. One useful choice is the "static" gauge

$$\lambda = X^0 \ (\equiv t), \tag{2.13}$$

which identifies the timelike coordinate of the world line with the time in the spacetime. In this gauge, with the flat Minkowski spacetime background $G_{MN}(x) = \eta_{MN} = \mathrm{diag}(-,+,\cdots,+)$ the action (2.11) becomes

$$S_0 = -m \int_{t_0}^{t_1} \mathrm{d}t \sqrt{1 - \vec{v}^2}, \tag{2.14}$$

where $\vec{v} \equiv \mathrm{d}\vec{X}/\mathrm{d}t$. If one goes to the nonrelativistic approximation $|\vec{v}| \ll 1 (= c)$, up to the second order, the action (2.14) gives the potential energy of the rest mass m and the usual kinetic energy $m\vec{v}^2/2$ with proper sign.

Another useful gauge fixing is

$$\frac{\mathrm{d}X^M}{\mathrm{d}\tau} \frac{\mathrm{d}X^N}{\mathrm{d}\tau} G_{MN} \equiv U^M U_M = -1, \tag{2.15}$$

where we identify τ as the proper time of the particle and introduce the velocity $U^M \equiv \mathrm{d}X^M/\mathrm{d}\tau$. In this gauge, the equation of motion for $X^M(\tau)$ could be the timelike geodesic equation:

$$0 = \frac{\mathrm{d}^2 X^M}{\mathrm{d}\tau^2} + \Gamma^M{}_{NL} \frac{\mathrm{d}X^N}{\mathrm{d}\tau} \frac{\mathrm{d}X^L}{\mathrm{d}\tau} = U^N \nabla_N U^M, \tag{2.16}$$

where $\Gamma^M{}_{NL}$ is the Christoffel symbol. In flat Minkowski spacetime or local inertial frames, where the Christoffel symbol vanishes, the motion simply becomes free fall with the mass-shell constraint (2.15). Thus, the Christoffel symbol determines how the acceleration could be modified from the straight line of the particle with the velocity.

2.2.1 Newtonian limit

We can consider Newton gravity by taking the Newtonian limit where the weak and static gravitation field and the nonrelativistic velocity $\mathrm{d}\vec{X}/\mathrm{d}\tau \ll \mathrm{d}t/\mathrm{d}\tau$ are required. We expand the static metric $(0 = \partial_t G_{MN}(x))$ around flat Minkowski spacetime

$$G_{MN}(x) = \eta_{MN} + h_{MN}(x) + \mathcal{O}(h^2). \tag{2.17}$$

Then, the geodesic equation (2.16) is reduced to be an equation of motion under a gravitation potential,

$$\frac{\mathrm{d}^2 \vec{X}}{\mathrm{d}t^2} = -\vec{\nabla} V(x), \tag{2.18}$$

where the proper time has been eliminated through the choice $\tau = t$. The Newton potential $V(x)$ is given by the deviation of the time-time component of the metric

$$V(x) = -\frac{1}{2} h_{tt}(x). \tag{2.19}$$

The Newton potential is determined by the Einstein equation (2.3), which becomes the Poisson equation in the Newtonian limit,

$$\Delta V(x) = 8\pi G_{\mathrm{N}}^{(D)} \frac{D-3}{D-2}\, \rho(x), \tag{2.20}$$

where $\rho(x) = T_{tt}(x)$ is the rest energy which is dominant in $T_{MN}(x)$ ($|T_{ij}| \ll |T_{tt}|$) and Δ is the Laplacian for $(D-1)$-dimensional spatial coordinates. For a point particle with a rest mass M at the origin i.e. $\rho(x) = M\delta^{D-1}(\vec{x})$, one can find the potential[3]

$$V(r) = -\frac{8\pi G_{\mathrm{N}}^{(D)} M}{(D-2)V_{S^{(D-2)}}}\frac{1}{r^{D-3}}, \tag{2.21}$$

where $r = |\vec{x}|$ is the radial "transverse" to the point particle, and V_{S^d} is the volume of d-dimensional unit sphere. Therefore, the presence of the rest point particle with a mass M modifies the flat metric as

$$G_{tt} = -1 + \frac{16\pi G_{\mathrm{N}}^{(D)} M}{(D-2)V_{S^{(D-2)}}}\frac{1}{r^{D-3}} + \mathcal{O}(1/r^{D-2}). \tag{2.22}$$

This can be used to observe the mass in asymptotically ($r \to \infty$) flat spacetime[4]. We define the Schwarzschild radius r_{S} as

$$r_{\mathrm{S}}^{D-3} = \frac{16\pi G_{\mathrm{N}}^{(D)} M}{(D-2)V_{S^{(D-2)}}} \tag{2.23}$$

which gives a characteristic length of spacetime. This also implies that the backreaction measured by the deviation from the flat metric can be observed at the scale of the order r_{S}.

This kind of deformation of the spacetime typically appears in black hole geometries. The most simplest example is the Schwarzschild black hole which is a vacuum solution of the Einstein equation:

$$\mathrm{d}s^2 = -\left(1 - \frac{r_{\mathrm{S}}^{D-3}}{r^{D-3}}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\left(1 - \frac{r_{\mathrm{S}}^{D-3}}{r^{D-3}}\right)} + r^2\mathrm{d}\Omega_{D-2}^2, \tag{2.24}$$

where r_{S} is given in (2.23) and M is the mass of Schwarzschild black hole. This is the static spherical solution and asymptotically flat. If one approaches to the small r region, one faces the event horizon located at $r = r_{\mathrm{S}}$.

Let us now discuss the gravitational red shift. In the curved spacetime, the emitted/observed frequencies of a signal in two different points are different. By using (2.15) with $X^M = (t, x^i)$, one observes the relation between the proper time interval $\mathrm{d}\tau$ and the coordinate time interval $\mathrm{d}t$ as $\mathrm{d}\tau = \sqrt{-G_{tt}(x)}\,\mathrm{d}t$, where we assumed that the clock is at rest with respect to the reference frame $\mathrm{d}x^i = 0$. Assuming that the gravitational field is stationary, we refer to the coordinate time t as the universal time. Therefore, the time intervals in two points, say the points x_1 and x_2 where the signal is emitted and observed, respectively, are the same $\Delta t_1 = \Delta t_2$. The emission frequency of the source defined by $\omega_1 \equiv 1/\Delta\tau_1$ is given as $\omega_1 = 1/(\sqrt{-G_{tt}(x_1)}\Delta t_1)$. Since the same is true for observer at x_2, one can conclude that

$$\omega_2 = \frac{\sqrt{-G_{tt}(x_1)}}{\sqrt{-G_{tt}(x_2)}}\,\omega_1. \tag{2.25}$$

Applying this to the Schwarzschild metric (2.24) at the points $r_1 < r_2$, one obtains the red shift $\omega_2 < \omega_1$ because of $|G_{tt}(x_2)| > |G_{tt}(x_1)|$. Moreover, if the source approaches the horizon $r_1 \to r_{\mathrm{S}}$, the observed frequency at r_2 tends to zero in the finite frequency ω_1. The horizon can be characterized as a surface of infinite red shift.

[3]In this normalization, we arrive at the usual form of 4D Newton gravity potential $V(r) = -G_{\mathrm{N}}^{(4)} M/r$.

[4]More precisely, we could define the mass in several formalisms such as ADM mass or Komar mass.

2.2.2 Electromagnetic interaction

Now let us introduce the electromagnetic field in the bulk spacetime. The action of the Maxwell gauge field $A_1(x) = A_M(x)\mathrm{d}x^M$ reads as usual,

$$S_{\mathrm{Maxwell}} = -\frac{1}{4}\int_{\mathcal{M}} \mathrm{d}^D x\sqrt{-G}F^{MN}F_{MN} = -\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^D x\sqrt{-G}|F_2|^2 = -\frac{1}{2}\int_{\mathcal{M}} F_2 \wedge *F_2, \tag{2.26}$$

where the field strength is given by $\frac{1}{2}F_{MN}\mathrm{d}x^M \wedge \mathrm{d}x^N = F_2 = \mathrm{d}A_1 = \frac{1}{2}\Big(\partial_M A_N - \partial_N A_M\Big)\mathrm{d}x^M \wedge \mathrm{d}x^N$. The Maxwell field in the bulk could couple to the particle world line as

$$S_{0\mathrm{WZ}} = -e\int_\gamma \mathrm{d}X^M A_M(X) = -e\int_\gamma \mathrm{d}\lambda \frac{\mathrm{d}X^M}{\mathrm{d}\lambda}A_M(X), \tag{2.27}$$

where e is the charge of the particle and we refer to this sort of term as Wess-Zumino term. The gauge invariance of the action (2.27) under the gauge transformation with a gauge parameter $\theta(x)$,

$$\delta A_M = \partial_M \theta\,, \tag{2.28}$$

requires the world line not to be terminated if there are no sources. The dynamics of the system is described by the actions (2.11), (2.26), and (2.27). In order to obtain the equation of motion for the Maxwell field $A_M(x)$, it is convenient to use the identity

$$A_M(X) = \int_{\mathcal{M}} \mathrm{d}^D x \delta^D(x - X(\lambda))A_M(x),$$

in which the delta function localizes the Maxwell field $A_M(x)$ to the world line. Varying the actions with respect to the Maxwell field $A_M(x)$, we obtain

$$0 = \nabla_M F^{MN} - J^N \quad \Longleftrightarrow \quad 0 = \mathrm{d} * F_2 - *J, \tag{2.29}$$

where the current $J^M(x)$ is defined by

$$J^M(x) \equiv \frac{e}{\sqrt{-G}}\int_\gamma \mathrm{d}\lambda \frac{\mathrm{d}X^M}{\mathrm{d}\lambda}\delta^D(x - X(\lambda)). \tag{2.30}$$

By using the equation of motion (2.29), we arrive at the continuity equation

$$0 = \nabla_M J^M \quad \Longleftrightarrow \quad 0 = \mathrm{d} * J. \tag{2.31}$$

The continuity equation can be also obtained by requiring the gauge invariance of the action (2.27).

One can introduce the conserved charge through the continuity equation (2.31),

$$Q = \int_{V_t} \mathrm{d}^{D-1}x\sqrt{-G}J^0 = \int_{V_t} *J, \tag{2.32}$$

where V_t is the volume at some time t. In the static gauge (2.13), it is easy to see that the charge Q is nothing but e. By using the equation of motion (2.29), we can rewrite the charge (2.32) in terms of the electric flux

$$Q = \int_{S^{(D-2)}} *F, \tag{2.33}$$

where we have assumed that the constant time slice has the boundary with topology $S^{(D-2)}$.

In the flat Minkowski spacetime with the static gauge, the action (2.27) is reduced to be

$$S_{0\mathrm{WZ}} = -e\int_\gamma \mathrm{d}t\Big(A_0(t,X^i) + \frac{\mathrm{d}X^i}{\mathrm{d}t}A_i(t,X^i)\Big). \tag{2.34}$$

Together with the free part of the action (2.14), we could obtain the equation of motion through the variation of X^i,

$$0 = m\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\frac{v_i}{\sqrt{1-\vec{v}^2}}\bigg) + e\Big(F_{i0} + F_{ij}v^j\Big). \tag{2.35}$$

The second term on the right correctly represents the electromagnetic forces acting on the charged particle.

2.2.3 Path-integral in flat background spacetime

We now quantize point particles in flat background spacetime. We first consider the case where the particle trajectory is given by the open path. For precise quantization, we depart from the nonlinear action (2.11) and consider the following on-shell equivalent action

$$\widetilde{S}_0[e,X] = -\frac{1}{2}\int_{\lambda_0}^{\lambda_1}\mathrm{d}\lambda\sqrt{-\gamma_{\lambda\lambda}}\Big(\gamma^{\lambda\lambda}\partial_\lambda X^M\partial_\lambda X^N\eta_{MN} + m^2\Big) = \frac{1}{2}\int_{\lambda_0}^{\lambda_1}\mathrm{d}\lambda\, e\Big(\frac{1}{e^2}\partial_\lambda X^M\partial_\lambda X_M - m^2\Big), \tag{2.36}$$

where we have introduced the world line metric (or the einbein) $\gamma_{\lambda\lambda}(\lambda)$ (or $e(\lambda) = \sqrt{-\gamma_{\lambda\lambda}(\lambda)}$). By using the equation of motion for the einbein $e(\lambda)$,

$$e^2 = -\frac{1}{m^2}g_{\lambda\lambda} = -\frac{1}{m^2}\partial_\lambda X^M\partial_\lambda X_M, \tag{2.37}$$

we could be back to the original action (2.11).

We now proceed to quantize the point particle through the path-integral formulation with the classical action (2.36) [20]. Since there is a one dimensional diffeomorphism invariance of the action;

$$\begin{aligned} &\lambda \to \lambda' = \lambda'(\lambda), \quad \text{with boundary conditions} \quad \lambda'(\lambda_0) = \lambda_0, \;\; \lambda'(\lambda_1) = \lambda_1, \\ \text{and} \quad &e(\lambda) \to e'(\lambda') = \left(\frac{\mathrm{d}\lambda}{\mathrm{d}\lambda'}\right)e(\lambda), \qquad X^M(\lambda) \to X'^M(\lambda') = X^M(\lambda), \end{aligned} \tag{2.38}$$

the path-integral should be evaluated over gauge inequivalent configurations. Therefore, the transition amplitude is formally given by

$$\langle X^M_{\mathrm{f}} = X^M(\lambda=1)|X^M_{\mathrm{i}} = X^M(\lambda=0)\rangle = \int\frac{\mathcal{D}e\,\mathcal{D}X}{V_{\mathrm{gauge}}}\,\mathrm{e}^{-\frac{1}{2}\int_0^1\mathrm{d}\lambda e\left(\frac{1}{e^2}(\partial_\lambda X^M)^2 + m^2\right)}, \tag{2.39}$$

where, for convenience, we have fixed the parameterization of the end points as $\lambda_0 = 0$ and $\lambda_1 = 1$ and taken the Wick rotation $\lambda \to -i\lambda$ and $X^0 \to -iX^0$. In (2.39), the formal gauge volume is denoted by V_{gauge} and we put it inside the integral. In order to define the integration measures and the gauge volume, we first introduce the reparameterization (gauge) invariant norm through the inner product of the tangent functional space,

$$||\delta e||_e^2 = \langle\delta e,\delta e\rangle_e \equiv \int_0^1\mathrm{d}\lambda\frac{1}{e}(\delta e)^2, \qquad ||\delta X||_e^2 = \langle\delta X,\delta X\rangle_e \equiv \int_0^1\mathrm{d}\lambda\, e(\delta X)^2. \tag{2.40}$$

These norms give some heuristic guiding principle to find the volume elements. In finite dimensional case, the Riemann structures with line element (2.1) induce the natural volume elements $\sqrt{|G(x)|}\,\mathrm{d}x^1\wedge$

$\cdots\wedge \mathrm{d}x^D$. We may be able to read off "functional metric" through the line element (2.40) and define the integration measure in the similar manner. What we should do first is to choose precise "coordinates system" in the functional space.

Infinitesimal gauge variation of (2.38) with $\delta\lambda(\lambda) = \lambda'(\lambda) - \lambda \equiv -\epsilon(\lambda)$ is given by

$$\delta_{\text{gauge}} e = \partial_\lambda(\epsilon e). \tag{2.41}$$

On the other hand, there exists physical variation $\delta_\perp e(\lambda)$ which could not be achieved through the gauge transformation (2.41). The orthogonality in terms of the inner product defined by (2.40) gives

$$0 = \int_0^1 \mathrm{d}\lambda \frac{1}{e}\big(\delta_{\text{gauge}} e\big)\big(\delta_\perp e\big) = -\int_0^1 \mathrm{d}\lambda \epsilon e \partial_\lambda\Big(\frac{1}{e}\delta_\perp e\Big), \quad \text{i.e.} \quad \delta_\perp e = \delta c e, \quad \text{with} \quad \delta c : \text{const.}, \tag{2.42}$$

where we have integrated by part and imposed the boundary conditions $\epsilon(\lambda = 0, 1) = 0$. Therefore, the general variation of the einbein is given by the orthogonal sum of the reparameterization $\delta_{\text{gauge}} e(\lambda)$ and the global scale transformation $\delta_\perp e(\lambda)$. It is easy to see the scale transformation defined in (2.42) could be related to the variation of the length of the world line l which is gauge invariant

$$(0 <)\ l = \int_0^1 \mathrm{d}\lambda e \quad \Longrightarrow \quad \delta l = \delta c\, l. \tag{2.43}$$

Plugging the variation $\delta e = \delta_{\text{gauge}} e + \delta_\perp e$ into the invariant norm (2.40), we obtain

$$||\delta e||_e^2 = \int_0^1 \mathrm{d}\lambda\, e^3 \epsilon\Big(-\frac{1}{e^2}\partial_\lambda \frac{1}{e}\partial_\lambda e\Big)\epsilon + \frac{\delta l^2}{l}, \tag{2.44}$$

where we have defined an invariant inner product of the gauge parameter

$$||\epsilon||_e^2 \equiv \int_0^1 \mathrm{d}\lambda\, e^3 \epsilon^2. \tag{2.45}$$

By using "one" gauge degrees of freedom $\epsilon(\lambda)$, we may fix the gauge as $e(\lambda) = l$ where the constant l should be the length of the world line defined in (2.43). The parameter l is called as the modular parameter. In order to estimate the volume forms through (2.44) and (2.45), we need to fix the topology of the world line, which affects the boundary conditions of the function $\epsilon(\lambda)$.

First, as before, we discuss the case of open path with the boundary condition $\epsilon(\lambda = 0, 1) = 0$. It is convenient to use eigenfunctions of the real operator $-\partial_\lambda^2$ in (2.44) which are given by $\xi_n(\lambda) = \sqrt{2}\sin(n\pi\lambda)$, $(n = 1, 2, \cdots)$. By using the linear combination $\epsilon(\lambda) = \sum_{n=1}^\infty \delta a_n \xi_n(\lambda)$, we can evaluate (2.44) and (2.45) as

$$||\delta e||_{e=l}^2 = \sum_{n=1}^\infty \pi^2 l n^2 \delta a_n^2 + \frac{\delta l^2}{l}, \qquad \text{and} \qquad ||\epsilon||_{e=l}^2 = \sum_{n=1}^\infty l^3 \delta a_n^2. \tag{2.46}$$

Thus, we could obtain the volume forms[5];

$$\mathcal{D}e = \Big(\prod_{n=1}^\infty \pi\sqrt{l}n\, \mathrm{d}a_n\Big)\frac{1}{\sqrt{l}}\mathrm{d}l = \sqrt{2}\mathcal{D}\epsilon\, \mathrm{d}l, \quad \text{with} \quad \mathcal{D}\epsilon = l^{-3/4}\prod_{n=1}^\infty \mathrm{d}a_n \equiv V_{\text{gauge}}. \tag{2.47}$$

[5]We here adopt the zeta function regularization for infinite products. Since the zeta function $\zeta(s)$ can be analytically continued to a meromorphic function of s in the complex s-plane, $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$ and $\zeta'(s) = -\sum_{n=1}^\infty \frac{\log n}{n^s}$ provide

$$1 + 1 + \cdots = \zeta(0) = -\frac{1}{2}, \qquad 1\cdot 2 \cdots = \mathrm{e}^{-\zeta'(0)} = \sqrt{2\pi}.$$

We decompose the functional degrees of freedom of the einbein $e(\lambda)$ to those for the gauge "coordinate" $\epsilon(\lambda)$ and for the modular parameter ("coordinate") l. Since the gauge volumes will be canceled out in (2.39), the amplitude has the finite integration over l.

Next, we shall do path-integral for $X^M(\lambda)$, decomposing the classical part and the fluctuation part $\widetilde{X}^M(\lambda)$:

$$X^M(\lambda) = X_{\rm i}^M + \big(X_{\rm f}^M - X_{\rm i}^M\big)\lambda + \widetilde{X}^M(\lambda), \tag{2.48}$$

where the boundary conditions $X^M(0) = X_{\rm i}^M$ and $X^M(1) = X_{\rm f}^M$ imply $\widetilde{X}^M(\lambda = 0,1) = 0$. Since the classical part fixed by the boundary conditions does not represent any degrees of freedom for the integration, we consider only the fluctuation part in the path-integration. Plugging (2.47) and (2.48) into (2.39) and using the boundary condition for $\widetilde{X}^M(\lambda)$, we get

$$\langle X_{\rm f}^M | X_{\rm i}^M \rangle \propto \int_0^\infty {\rm d}l \, {\rm e}^{-\frac{1}{2}\left(\frac{1}{l}(X_{\rm f}-X_{\rm i})^2 + lm^2\right)} \int \mathcal{D}\widetilde{X} \, {\rm e}^{-\frac{1}{2}\int_0^1 {\rm d}\lambda \widetilde{X}^M\left(-\frac{1}{l}\partial_\lambda^2 \widetilde{X}_M\right)}. \tag{2.49}$$

Following the evaluation of the integration measure of the vielbein, we expand the fluctuation as $\widetilde{X}^M(\lambda) = \sum_{n=1}^\infty \widetilde{x}_n^M \sqrt{2}\sin(n\pi\lambda)$. Then the integration of the fluctuation can be performed as

$$\int \mathcal{D}\widetilde{X} \, {\rm e}^{-\frac{1}{2}\int_0^1 {\rm d}\lambda \widetilde{X}^M\left(-\frac{1}{l}\partial_\lambda^2 \widetilde{X}_M\right)} = \left(\int \Big(\prod_{n=1}^\infty \sqrt{l}{\rm d}\widetilde{x}_n\Big) \, {\rm e}^{-\frac{1}{2}\sum_{n=1}^\infty \frac{\pi^2}{l} n^2 \widetilde{x}_n \widetilde{x}_n}\right)^D = \left(\frac{\pi}{2}\right)^{D/4} (2\pi l)^{-D/2}. \tag{2.50}$$

Therefore, we could organize

$$\begin{aligned}\langle X_{\rm f}^M | X_{\rm i}^M \rangle &\propto \int_0^\infty {\rm d}l \, (2\pi l)^{-D/2} {\rm e}^{-\frac{1}{2}\left(\frac{1}{l}(X_{\rm f}-X_{\rm i}) + lm^2\right)} = \int_0^\infty {\rm d}l \int \frac{{\rm d}^D p}{(2\pi)^D} \, {\rm e}^{ip(X_{\rm f}-X_{\rm i}) - \frac{l}{2}(p^2+m^2)} \\ &= \int \frac{{\rm d}^D p}{(2\pi)^D} \, {\rm e}^{ip(X_{\rm f}-X_{\rm i})} \int_0^\infty {\rm d}l \, {\rm e}^{-lH} = \int \frac{{\rm d}^D p}{(2\pi)^D} \frac{{\rm e}^{ip(X_f - X_i)}}{H},\end{aligned} \tag{2.51}$$

which is nothing but Feynman propagator for the free relativistic scalar field in D-dimensions. In the expression (2.51), we have defined the Hamiltonian $H = (p^2 + m^2)/2$ so that the parameter l corresponds to Schwinger parameter.

Let us briefly consider the case of closed path i.e. one-loop vacuum amplitude. Due to the periodic boundary condition $\epsilon(\lambda) = \epsilon(\lambda+1)$, the eigenfunction may be given by $\epsilon_n(\lambda) = {\rm e}^{i2\pi n\lambda}$, $(n = 0, \pm 1, \cdots)$. Comparing the open path, the constant zero mode which corresponds to the translation in the circle appears. The same procedure gives the measures as

$$\mathcal{D}e = \Big(\prod_{n\neq 0} {\rm d}a_n\Big)\frac{{\rm d}l}{l}, \qquad \mathcal{D}\epsilon = \prod_n {\rm d}a_n = \Big(\prod_{n\neq 0} {\rm d}a_n\Big) L \equiv V_{\rm gauge}, \tag{2.52}$$

where we formally replaced the integration of the zero mode by L. Then, the one-loop amplitude is given by

$$Z_{S^1} \propto \int_0^\infty \frac{{\rm d}l}{2l} \, {\rm e}^{-\frac{1}{2}lm^2} \int \mathcal{D}X \, {\rm e}^{-\frac{1}{2}\int_0^1 {\rm d}\lambda X^M\left(-\frac{1}{l}\partial_\lambda^2 X_M\right)} = \int_0^\infty \frac{{\rm d}l}{2l} \, {\rm e}^{-\frac{1}{2}lm^2} (2\pi l)^{-D/2} = \int \frac{{\rm d}^D p}{(2\pi)^D} \int_0^\infty \frac{{\rm d}l}{2l} \, {\rm e}^{-lH}. \tag{2.53}$$

It should be mentioned that in any D-dimensional spacetime, in UV regime where $l \to 0$, the amplitude diverges. On the other hand, in IR, the amplitude converges as long as $m^2 > 0$. We will compare this result of particle one-loop amplitude with string one-loop.

2.3 p-brane

So far we have discussed the motion of a particle in a general background spacetime, the coupling to the external electromagnetic fields, and the path-integral quantization. One could generalize the world line description (2.11), (2.12), (2.27) and (2.39) to that for a p-dimensional extended object. The parameterization of this object is now given by σ^m ($m = 0, 1, \cdots, p$) and the embedding is given through the target spacetime (bulk) coordinates $X^M(\sigma)$. The action may be proportional to the volume swept by the p-brane. The invariant world volume, which is a natural generalization of the world line (2.11), is given by

$$S_p = -T_p \times (\text{volume}) = -T_p \int_\Sigma \mathrm{d}^{1+p}\sigma \sqrt{-\det g_{mn}(\sigma)}, \tag{2.54}$$

where the induced metric on the world volume is now

$$g_{mn}\big(X(\sigma)\big) = \partial_m X^M \partial_n X^N G_{MN}(X)\,. \tag{2.55}$$

The determinant inside the square root is for the indices of the world volume labeling m and n. T_p is the tension of the p-brane whose mass dimension is mass^{1+p}.

Let us analyze the interaction between p-brane and other spacetime fields. Here we consider a $(p+1)$-form gauge potential $A_{p+1}(x)$, which is a natural generalization of the usual Maxwell field $A_1(x) = A_M(x)\mathrm{d}x^M$:

$$A_{p+1}(x) \equiv \frac{1}{(p+1)!} A_{M_1\cdots M_{p+1}}(x)\mathrm{d}x^{M_1} \wedge \cdots \wedge \mathrm{d}x^{M_{p+1}}, \tag{2.56}$$

where $A_{M_1\cdots M_{p+1}}(x)$ is a totally antisymmetric tensor field. A $(p+2)$-form field strength $F_{M_1\cdots M_{p+2}}(x)$ can be defined by $F_{p+2}(x) \equiv \mathrm{d}A_{p+1}(x)$. The field strength is invariant under the gauge transformation

$$\delta A_{p+1}(x) = \mathrm{d}\theta_p(x), \tag{2.57}$$

where $\theta_p(x)$ is a p-form gauge parameter. The kinetic term of the $(p+1)$-form field, which is the natural analog of (2.26), is given by

$$S_{(p+1)-\text{form}} = -\frac{1}{2}\int_{\mathcal{M}} F_{(p+2)} \wedge * F_{(p+2)} = -\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^D x \sqrt{-G}|F_{p+2}|^2. \tag{2.58}$$

We could introduce the minimal coupling between the $(p+1)$-form gauge field and the p-brane, which is the extended version of (2.27),

$$\begin{aligned} S_{p\mathrm{WZ}} &= -q_p \int_\Sigma A_{p+1} = -q_p \int_\Sigma \frac{1}{(p+1)!} A_{M_1\cdots M_{p+1}}(X)\mathrm{d}X^{M_1} \wedge \cdots \wedge \mathrm{d}X^{M_{p+1}} \\ &= -q_p \int_\Sigma \frac{1}{(p+1)!} A_{M_1\cdots M_{p+1}}(X)(\partial_{m_1}X^{M_1})\cdots(\partial_{m_{p+1}}X^{M_{p+1}})\mathrm{d}\sigma^{m_1} \wedge \cdots \wedge \mathrm{d}\sigma^{m_{p+1}}, \end{aligned} \tag{2.59}$$

where q_p is the charge of the p-form field and the support of the integration is the $(p+1)$-dimensional world volume. The gauge invariance tells us that the p-brane should not have the boundary. However if the boundary is attached to other branes, one could define the charge.

The equation of motion for the gauge field is

$$0 = \nabla_M F^{MN_1\cdots N_{p+1}} - J^{N_1\cdots N_{p+1}} \qquad \Longleftrightarrow \qquad 0 = \mathrm{d} * F_{p+2} - *J_{p+1}, \tag{2.60}$$

where

$$J^{M_1\cdots M_{p+1}}(x) = \frac{q_p}{\sqrt{-G}} \int_\Sigma \mathrm{d}\sigma^{m_1} \wedge \cdots \wedge \mathrm{d}\sigma^{m_{p+1}} (\partial_{m_1}X^{M_1})\cdots(\partial_{m_{p+1}}X^{M_{p+1}})\delta^D(x - X(\sigma)). \tag{2.61}$$

The continuity equation

$$0 = \nabla_M J^{MN_1\cdots N_p} \quad \Longleftrightarrow \quad 0 = d * J_{p+1} \tag{2.62}$$

implies a p-brane charge q_p

$$q_p = \int_{B^{(D-p-1)}} *J_{p+1} = \int_{S^{(D-p-2)}} *F_{p+2}, \tag{2.63}$$

where $\partial B^{D-p-1} = S^{D-p-2}$ and we have used the equation of motion (2.60) and the Stokes' theorem.

Once the field strength F_{p+2} (electric) is defined, one can always consider its dual $\widetilde{F}_{D-p-2}$ (magnetic) via Hodge dual operation,

$$*F_{p+2} \equiv \widetilde{F}_{D-p-2} = \mathrm{d}\widetilde{A}_{D-p-3}.$$

The dual gauge field $\widetilde{A}_{D-p-3}(x)$ may couple to the $(D-p-4)$-brane, so that the p-brane and the $(D-p-4)$-brane are dual to each other. It has been shown that the charges should satisfy the Dirac quantization condition [21]

$$q_p q_{D-p-4} = 2\pi n, \qquad n \in Z. \tag{2.64}$$

2.4 String

We here consider the case $p = 1$, i.e. string. Comparing with the other extended objects, string with 2D world sheet is special. In flat background spacetime, one could quantize the string itself and analyze physics in the target spacetime through the 2D world sheet dynamics by using well-developed CFT and 2D quantum gravity.

String theory appeared originally through attempts to explain the hadron resonance spectrum of the strong interaction. The Veneziano scattering amplitude [22] described the dynamics of relativistic open string. However, experiments in those days turned out to support the gauge theory i.e. QCD instead of string. On the other hand, relativistic closed strings have a massless spin 2 particle in their spectrum, which precisely corresponds to the graviton. The close string theory turned to be the theory of the gravity [23]. This is the initial place for string theory as a unified theory.

Possible shapes of this one-dimensional object are line and loop which are referred as an open string and a closed string, respectively[6]. The length of string depends on the physics under consideration. When the string theory is describing quantum gravity, the typical size of string is the Planck length l_{P}. Much below the string scale, there is not enough resolution to distinguish string to point particle. However, it contains various oscillation modes which correspond to various quantum numbers. The energies and the polarizations of the oscillation modes are related to the masses and the spins of the corresponding elementary particles.

2.4.1 String world sheet

For strings the action (2.54) is nothing but the Nambu-Goto action for the area of the world sheet parameterized by the coordinates $\sigma^m = (\tau, \sigma)$ with $0 \le \sigma \le l$ for the spatial coordinate:

$$S_1[X] = -\frac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\sqrt{-\det g_{mn}(\sigma)}. \tag{2.65}$$

The string length l_{s} is defined by $l_{\mathrm{s}}^2 = \alpha'$, which is the only dimensionful external parameter in string theories[7]. Closed string provides the world sheet without boundaries, while open string sweeps out world sheet with boundaries.

[6]We consider only orientable world sheets.

[7]In 2D, such a dimensionful coupling constant is not irrelevant but marginal. The world sheet theory does not suffer from UV divergences.

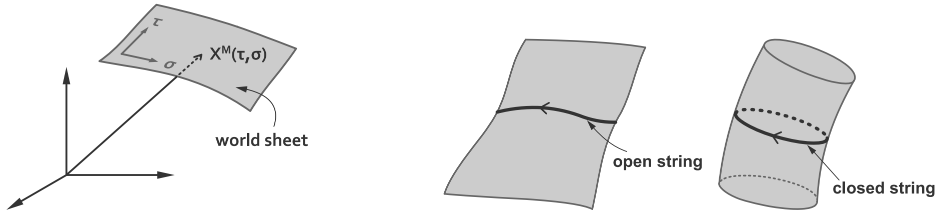

Figure 2: String world sheets

In quantum theory, it is not convenient to use the action (2.65) mainly because of its high nonlinearity of the square root. As we discussed in the case of particles, we move on to the Polyakov action which provides on-shell equivalent description of the original Nambu-Goto action:

$$\widetilde{S}_1[X,\gamma] = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\sqrt{-\gamma}\gamma^{mn}\partial_m X^M \partial_n X^N G_{MN}(X), \tag{2.66}$$

where the auxiliary field $\gamma_{mn}(\sigma)$, which can be understood as the metric on the world sheet, has been introduced[8]. The action (2.66) now acquires an additional local symmetry, i.e. Weyl symmetry (local scale symmetry) $\gamma_{mn}(\sigma) \to \mathrm{e}^{2\lambda(\sigma)}\gamma_{mn}(\sigma)$, which plays a central role in string theories. As we will see, this Weyl rescaling may be used to reshape awkward diagrams to standard forms which are easy to deal with. Together with the general coordinate transformations, we could fix the 2D world sheet metric to the conformal gauge $\gamma_{mn}(\sigma) = \eta_{mn}$, at least locally. In a usual world sheet with nontrivial topologies, the conformal gauge cannot be imposed globally. Note that the conformal gauge cannot fix the gauges completely. There exists residual symmetry which is conformal symmetry.

The action (2.66) is the nonlinear sigma model of 2D scalar fields $X^M(\sigma)$ in the curved background spacetime whose metric is given by $G_{MN}(X)$. We will analyze the world sheet action (2.66) by using 2D quantum field theory. From the point of view of renormalizability, the action (2.66) is not complete. We will later find other pieces to complete the full nonlinear sigma model action.

By using the equation of motion of $\gamma_{mn}(\sigma)$, which is the energy momentum tensor in the 2D world sheet,

$$0 = T_{mn} \equiv \frac{2}{\sqrt{-\gamma}}\frac{\delta\widetilde{S}_1}{\delta\gamma^{mn}} = -\frac{1}{2\pi\alpha'}\Big(g_{mn} - \frac{1}{2}\gamma_{mn}\gamma^{pl}g_{pl}\Big), \tag{2.67}$$

and eliminating the auxiliary field $\gamma_{mn}(\sigma)$ through the algebraic equation above, one could be back to the original (2.65). The energy momentum tensor is traceless $0 = \gamma^{mn}T_{mn}$ which reflects that the theory is classically scale invariant.

The variation of the action (2.66) with respect to $X^M(\sigma)$ reads

$$\begin{aligned}\delta\widetilde{S}_1[X,\gamma] = {}& \frac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\sqrt{-\gamma}\Big(\nabla^2 X^M + \gamma^{mn}\partial_m X^L \partial_n X^P \Gamma^M{}_{LP}\Big)G_{MN}\delta X^N \\ & -\frac{1}{2\pi\alpha'}\int_{\partial\Sigma}(\mathrm{d}\Sigma)^m \partial_m X^M G_{MN}\delta X^N. \end{aligned} \tag{2.68}$$

The first line gives the equation of motion for $X^M(\sigma)$, and the second line is the boundary term. For closed string, the boundary term simply vanishes, while that is not the case for the open string. We need to impose boundary conditions at the endpoints of the open string. These are Neumann boundary condition and Dirichlet boundary condition, respectively:

$$n^m\partial_m X^M\Big|_{\partial\Sigma} = 0, \qquad X^M\Big|_{\partial\Sigma} = c^M, \tag{2.69}$$

[8] The metric $\gamma_{mn}(\sigma)$ is conceptually different from the induced metric $g_{mn}(\sigma)$.

where n^m is the unit normal vector to the boundary and c^M is the constant which describes the position of fixed endpoints of the open string. It should be noted that these boundary conditions preserve the local scale symmetry in the world sheet. Since the momentum conservation breaks down at the string endpoints with Dirichlet boundary conditions, the string should be attached to another extended object, which should be dynamical. As we will see, the Dirichlet boundary condition could not be imposed in the string perturbation in the vacuum, but in a solitonic background.

The boundary condition (2.69) can be imposed in each spacetime direction and string endpoint, independently. For simplicity, we impose the same boundary conditions on each direction; Neumann boundary conditions for directions ($M = m = 0, 1, \cdots, p$) and Dirichlet boundary conditions for the others ($M = a = p+1, \cdots, D-1$). In this case, the string endpoints make a single hypersurface in the target spacetime. This hypersurface can be interpreted as the world volume of p-dimensional object. This is the Dirichlet p-brane, in short Dp-brane.

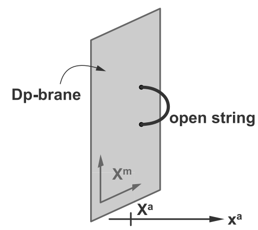

Figure 3: Dp-brane

Let us consider the partition function in the path-integral formulation:

$$Z = \int \frac{\mathcal{D}X\mathcal{D}\gamma}{V_{\text{gauge}}}\, \mathrm{e}^{-\widetilde{S}_1[X,\gamma]}. \tag{2.70}$$

As we have demonstrated in the case of particles, we have to sum up all possible gauge inequivalent physical fluctuations on the world sheet, where the gauge degrees of freedom are those for diffeomorphism and Weyl rescaling. Physically, this summation corresponds to that for all possible string excitations which are equivalent to the particle states.

We now perturb the bulk metric from the flat Minkowski spacetime,

$$G_{MN}(X) = \eta_{MN} + h_{MN}(X). \tag{2.71}$$

Apart from the kinetic term in the flat Minkowski background, the action (2.66) produces the interaction term between string and the external field which is called the vertex operator (external source term),

$$V_h \equiv \frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\sqrt{-\gamma}\gamma^{mn}\partial_m X^M \partial_n X^N h_{MN}(X). \tag{2.72}$$

In the path-integral formulation, this corresponds to the local operator insertion to the world sheet

$$Z = \int \frac{\mathcal{D}X\mathcal{D}\gamma}{V_{\text{gauge}}}\, \mathrm{e}^{-\widetilde{S}_1|_{\text{free}}}\Big(1 - V_h + \frac{1}{2!}V_h^2 + \cdots\Big). \tag{2.73}$$

This can be understood from the following world sheet point of view. Consider first a closed string propagating in bulk and interacting with a string world sheet (left side of Fig.4). With the Weyl rescaling, the string propagator may be replaced by the point on the world sheet. This injection is nothing but the insertion of the vertex operator. The vertex operator corresponds to emission/absorption of a string in a particular mass eigenstate, that is one of the various modes which exist in initial string states. If one takes the perturbation as the plane wave i.e.

$$h_{MN}(X) = \zeta_{MN}\, \mathrm{e}^{ik\cdot X}, \tag{2.74}$$

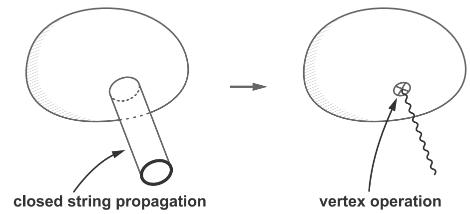

Figure 4: String interaction and vertex operator

and imposes the local scale invariance even in quantum theory, the operator is restricted as

$$k^2 = 0, \qquad \zeta_{MN} k^N = \zeta_{MN} k^M = 0. \tag{2.75}$$

These massless and transverse polarization conditions imply that the vertex operator (2.72) describes a emission/absorption of graviton. Instead of this S-matrix calculation, we could consider the general source $G_{MN}(X)$. As we will see in the following sections, again requiring the Weyl invariance of the world sheet, the spacetime metric $G_{MN}(X)$ turns out to follow the Einstein equation.

2.4.2 Strings in general background

We now introduce another renormalizable term which respects the Weyl symmetry in the world sheet:

$$S_{\Phi_0} = \Phi_0 \left(\frac{1}{4\pi} \int_\Sigma \mathrm{d}^2\sigma \sqrt{-\gamma} R(\gamma) + \frac{1}{2\pi} \int_{\partial\Sigma} \mathrm{d}s K \right), \quad \Phi_0 : \text{const.}, \tag{2.76}$$

where $R(\gamma)$ is the scalar curvature of $\gamma_{mn}(\sigma)$ and K is the extrinsic (geodesic) curvature of the boundary. This is the Einstein-Hilbert action in 2D spacetime with boundary. One could freely add this to the Polyakov action (2.66) without changing the equation of motion for the world sheet metric $\gamma_{mn}(\sigma)$ (2.67). Indeed, Riemann-Roch theorem tells that the parenthesis in (2.76) gives topological invariant of the world sheet (Euler number χ),

$$\frac{1}{4\pi} \int_\Sigma \mathrm{d}^2\sigma \sqrt{-\gamma} R(\gamma) + \frac{1}{2\pi} \int_{\partial\Sigma} \mathrm{d}s K = \chi = 2 - 2g - b, \tag{2.77}$$

where g and b are the number of handles and boundaries of the Riemann surface, respectively.

This world sheet topology is important in string interactions. Indeed, in the path-integral formulation, the summations should include the world sheet topologies. For point particles, the path-integral sums over all paths connecting the initial and the final states. For strings, one could naturally generalize this to summing over all world sheets, i.e. all embedding $X^M(\sigma)$ and all world sheet metric $\gamma_{mn}(\sigma)$, connecting the initial and the final strings (see Fig.5).

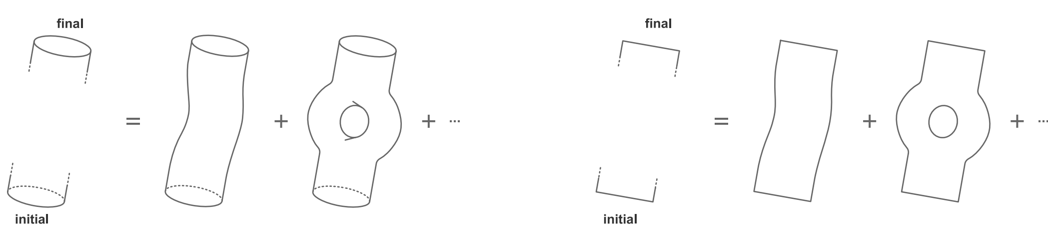

Figure 5: Genus expansions for closed string (left) and open string (right)

The string partition function may be modified by adding S_{Φ_0}:

$$Z = \int \frac{\mathcal{D}X\mathcal{D}\gamma}{V_{\text{gauge}}} \, \mathrm{e}^{-\widetilde{S}_1 - S_{\Phi_0}} = \sum_T (\mathrm{e}^{\Phi_0})^{-\chi(T)} \int_{\Sigma_T} \frac{\mathcal{D}X\mathcal{D}\gamma}{V_{\text{gauge}}} \, \mathrm{e}^{-\widetilde{S}_1}, \tag{2.78}$$

where the summation over metric can be decomposed with fixed topology T. The factor $(\mathrm{e}^{\Phi_0})^{-\chi(T)}$ gives the relative weight of different topologies. For a closed string case, if a handle is added in the world sheet, one gets a factor $\mathrm{e}^{2\Phi_0}$ through the Euler number. On the other hand, adding the handle can be regarded as the process for emitting and reabsorbing a closed string, i.e. two events of string interactions. Therefore one could identify a closed string coupling g_{closed} as $g_{\mathrm{closed}} = \mathrm{e}^{\Phi_0}$. Similar consideration could be applied for open string world sheets with handles and boundaries. As a result, open string coupling g_{open} is the square root of the closed string coupling. We then define the string coupling constant g_{s} as

$$g_{\mathrm{s}} \equiv \mathrm{e}^{\Phi_0} = g_{\mathrm{closed}} = g_{\mathrm{open}}^2. \tag{2.79}$$

For $g_{\mathrm{s}} \ll 1$, the summation over topologies defines a perturbative expansion, the genus expansion. The genus is the quantum string loop in the target spacetime, analogous to particle loops in quantum field theory. The dimensionless constant Φ_0 can be understood as the vacuum expectation value of the dilaton field, one of the massless spectra in string theories. In string theories, there are no dimensionless coupling constants, though the dimensionless coupling constants may be provided dynamically through the vacuum expectation values of dynamical fields, most probably nonperturbatively. So one may generalize the action (2.76) to

$$S_\Phi = \frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\sqrt{-\gamma}R(\gamma)\Phi(X) + \frac{1}{2\pi}\int_{\partial\Sigma}\mathrm{d}sK\Phi(X), \tag{2.80}$$

and define the constant Φ_0 as the asymptotic value of the dilaton field $\Phi_0 = \Phi(X = \infty)$. Setting a fluctuation part of the dilaton as $\widetilde{\Phi}(X) = \Phi(X) - \Phi_0$, we define the effective dynamical coupling $g_{\mathrm{s}}^{\mathrm{eff}}(X) = \mathrm{e}^{\Phi(X)} = g_{\mathrm{s}}\mathrm{e}^{\widetilde{\Phi}(X)}$. The perturbation now makes sense for $g_{\mathrm{s}}^{\mathrm{eff}}(X) \ll 1$. Due to the generic dilaton $\Phi(X)$, the action (2.80) itself looses Weyl invariance. However, the symmetry could be restored through the quantum effects coming from other Weyl invariant actions.

One could find another action which preserves the Weyl invariance in the string world sheet. Since the string is a one-dimensional extended object, it naturally couples to a two-form external gauge potential $B_2(X)$ through the action (2.59),

$$S_{1\mathrm{WZ}} = \frac{1}{4\pi\alpha'}\int_\Sigma B_2 = \frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\sqrt{-\gamma}\epsilon^{mn}\partial_m X^M\partial_n X^N B_{MN}(X), \tag{2.81}$$

where $\epsilon^{mn}(\sigma)$ is the Levi-Civita tensor in the 2D world sheet. We here just follow the convention of the charge $q_1 = 1/(4\pi\alpha')$. Since the action (2.81) does not depend on the 2D metric, that is Weyl invariant automatically, it does not change the equation (2.67). As we discussed in the previous section, the gauge invariance requires that the world sheet should not have any boundary, i.e. this interaction is only for closed strings. For open string world sheet, we need to perform the gauge transformation on the gauge field coupled to the boundary.

2.4.3 String equations and low energy effective theory

Now, we have the world sheet actions (2.66), (2.80), and (2.81), which may respect Weyl symmetry. From the target spacetime point of view, these describe the string in the external background fields $G_{MN}(X)$, $B_{MN}(X)$, and $\Phi(X)$, which are the lightest modes in string theories[9]:

$$S_\sigma = \frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\sqrt{\gamma}\Big\{\gamma^{mn}\partial_m X^M\partial_n X^N G_{MN}(X) + i\epsilon^{mn}\partial_m X^M\partial_n X^N B_{MN}(X) + \alpha' R(\gamma)\Phi(X)\Big\}. \tag{2.82}$$

[9]For completeness, we provide an action for background tachyon field $T(X)$ which couples to 2D cosmological constant term,

$$S_\sigma^T = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\sqrt{\gamma}T(X).$$

Here we adopted the Euclidean signature and considered only the closed string world sheet. In general, it is difficult to exactly solve this interacting world sheet theory in a general background on which strings propagate. However, if the background is weakly curved, we could treat the world sheet interactions perturbatively. In this case, in order to analyze the world sheet sigma model (2.82), one could use the background field method. We first expand the coordinates $X^M(\sigma)$ around a classical solution X_0^M,

$$X^M(\sigma) = X_0^M + \sqrt{\alpha'} Y^M(\sigma), \tag{2.83}$$

where $Y^M(\sigma)$ is chosen to be dimensionless. One can expand the action (2.82) and perform 2D world sheet field theory for the fluctuations $Y^M(\sigma)$. Expansions of the external fields $G_{MN}(X)$, $B_{MM}(X)$, and $\Phi(X)$ around X_0^M generate infinitely many spacetime derivative terms which can be regarded as couplings to the fields $Y^M(\sigma)$. Since this derivative expansion can be taken as an expansion in the powers of $(\sqrt{\alpha'}/r_{\rm S}) = (l_{\rm s}/r_{\rm S})$ with the characteristic length scale $r_{\rm S} = (\partial G/\partial X)^{-1}$ of the target spacetime, in the perturbation theory, we should impose

$$\frac{\sqrt{\alpha'}}{r_{\rm S}} \ll 1. \tag{2.84}$$

Therefore, these expansions are valid when the string length $\sqrt{\alpha'} = l_{\rm s}$ is too small to feel the curvature of the target spacetime. The string could be regarded as the point particle here. In this case, as in the conventional quantum field theory, starting from the free theory, we compute deviations order by order with the expansion parameter $(\sqrt{\alpha'}/r_{\rm S})$.

In the quantized string theories, we would like to keep the world sheet Weyl invariant. This puts some restrictions on the target spacetime. One can calculate the trace of energy momentum tensor in one-loop for the sigma model and the tree level of the string perturbation,

$$\langle T_m{}^m \rangle = -\frac{1}{2\alpha'}\beta^G_{MN}\gamma^{mn}\partial_m X^M \partial_n X^N - \frac{i}{2\alpha'}\beta^B_{MN}\epsilon^{mn}\partial_m X^M \partial_n X^N - \frac{1}{2}\beta^\Phi R, \tag{2.85}$$

and require that this should vanish. The beta functionals can be calculated as

$$\begin{aligned}
\beta^G_{MN} &= \alpha'\Big(R_{MN} + 2\nabla_M\nabla_N\Phi - \frac{1}{4}H_{MKL}H_N{}^{KL}\Big) + \mathcal{O}(\alpha'^2), \\
\beta^B_{MN} &= \alpha'\Big(-\frac{1}{2}\nabla^K H_{KMN} + (\partial^K\Phi)H_{KMN}\Big) + \mathcal{O}(\alpha'^2), \\
\beta^\Phi &= \frac{(D-D_{\rm c})}{6} + \alpha'\Big(-\frac{1}{2}\nabla^2\Phi + (\partial_M\Phi)^2 - \frac{1}{24}H_{KMN}H^{KMN}\Big) + \mathcal{O}(\alpha'^2),
\end{aligned} \tag{2.86}$$

where H_{MNL} is the field strength of two-form gauge field B_{MN}, and $D_{\rm c}$ is a contribution from Weyl anomaly, which is 26 for the bosonic string. Equations $\beta^G_{MN} = \beta^B_{MN} = \beta^\Phi = 0$ can be given as the equation of motion of the following action, which is the low energy effective action

$$S^{\rm eff}_{\rm closed} = \frac{1}{2\kappa_0^2}\int_{\mathcal{M}} {\rm d}^D x\sqrt{-G}\,{\rm e}^{-2\Phi}\Big\{R + 4(\partial_M\Phi)^2 - \frac{1}{12}H_{MNL}H^{MNL} - \frac{2(D-D_{\rm c})}{3\alpha'} + \mathcal{O}(\alpha')\Big\}, \tag{2.87}$$

where the coupling constant κ_0 will be determined later. The factor ${\rm e}^{-2\Phi}$ indicates a tree level contribution from the string perturbation. It should be mentioned that the effective action (2.87) can be

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This term leads to an effective action which may be added to (2.87),

$$S^{\rm eff}_{\rm tachyon} = \frac{1}{2\kappa_0^2}\int_{\mathcal{M}} {\rm d}^D x\sqrt{-G}\,{\rm e}^{-2\Phi}\Big\{-(\partial_M T)^2 + \frac{4}{\alpha'}T^2\Big\}.$$

However, since in superstring theories the tachyon could be projected out, we here simply neglect this effective action.

perfectly reproduced a priori through a different way via the string S-matrix computation. The solutions of the equations of motion describe backgrounds for string theory where strings could be consistently quantized in the lowest order in α' and the string coupling constant.

One solution of the equations of motion is $R_{MN} = B_{MN} = 0$, $\Phi = \Phi^0$ (const.) and $D = D_{\rm c} = 26$ which defines the critical dimensions of bosonic string. In the next subsection, we discuss the quantization of the bosonic string.

So far we have discussed the equation of motion for the closed string from the viewpoint of the world sheet nonlinear sigma model. Let us consider the same thing for open string in the closed string background where the closed string excitations are needed for consistent open string interactions. Since the endpoint of open string can be charged like a point particle, the spacetime gauge field naturally couples to the 1D boundary of the open string world sheet:

$$S_{\rm boundary} = \int_{\partial\Sigma} A_1 = \int_{\partial\Sigma} {\rm d}\Sigma^m \partial_m X^M A_M(X). \tag{2.88}$$

Here we assume that the Neumann boundary conditions are imposed in all directions. Together with the sigma model action (2.82) and the boundary term in (2.80), the beta functional for the coupling A_M could be evaluated. The equation of motion obtained by imposing the vanishing of the anomaly could be derived from the low energy effective action which is known as the Dirac-Born-Infeld action [24],

$$S^{\rm eff}_{\rm open} = -T_{D-1} \int_{\mathcal{M}} {\rm d}^D x \ {\rm e}^{-\Phi} \sqrt{-\det(G_{MN} + B_{MN} + 2\pi\alpha' F_{MN})}. \tag{2.89}$$

2.4.4 Quantization of bosonic string in flat background

We start with the Polyakov action in the flat background,

$$\widetilde{S}_1[X,\gamma] = -\frac{1}{4\pi\alpha'} \int {\rm d}^2\sigma \sqrt{-\gamma}\gamma^{mn} \partial_m X^M \partial_n X^N \eta_{MN}. \tag{2.90}$$

With the conformal gauge for 2D world-sheet $\gamma_{mn} = \eta_{mn} = {\rm diag}(-,+)$ the action (2.90) reduces to

$$S_{\rm B} = \frac{1}{\pi\alpha'} \int {\rm d}^2\sigma \partial_+ X^M \partial_- X_M, \tag{2.91}$$

where we have used light-cone coordinates on the world-sheet $\sigma^\pm \equiv \tau \pm \sigma$. The equation of motion of string coordinates turns to be free wave equation,

$$0 = (-\partial_\tau^2 + \partial_\sigma^2) X^M = \partial_+ \partial_- X^M. \tag{2.92}$$

When one imposes the conformal gauge, the equation of motion for γ_{mn} becomes a constraint,

$$0 = T_{mn}, \quad \text{i.e.} \quad 0 = \partial_+ X^M \partial_+ X_M = \partial_- X^M \partial_- X_M. \tag{2.93}$$

This constraint is the first class constraint which could generate gauge symmetry. This is nothing but conformal symmetry. The wave equation (2.92) can be solved by the sum of left and right movers which travel in the opposite directions along the string:

$$X^M(\tau,\sigma) = X^M_{\rm L}(\sigma^+) + X^M_{\rm R}(\sigma^-). \tag{2.94}$$

For closed string, imposing the periodic boundary condition $X^M(\tau,\sigma) = X^M(\tau,\sigma + l)$, one can get

$$X^M_{\rm L}(\sigma^+) = \frac{1}{2}x^M + \frac{\pi\alpha'}{l} p^M \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha^M_n}{n} {\rm e}^{-i\frac{2\pi}{l} n\sigma^+}, \tag{2.95a}$$

$$X^M_{\rm R}(\sigma^-) = \frac{1}{2}x^M + \frac{\pi\alpha'}{l} p^M \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\widetilde{\alpha}^M_n}{n} {\rm e}^{-i\frac{2\pi}{l} n\sigma^-}, \tag{2.95b}$$

where the zero modes, x^M and p^M, express the position and momentum of center of mass, respectively. Defining the canonical momentum $\Pi^M(\tau,\sigma) = \delta\widetilde{S}_1/\delta(\partial_\tau X_M)$, the Hamiltonian is given by

$$H = \frac{1}{2\pi\alpha'}\int_0^l \mathrm{d}\sigma\big((\partial_+ X)^2 + (\partial_- X)^2\big) = \frac{\pi}{l}\Big(\alpha' p^2 + \sum_{n\neq 0}\big(\alpha_{-n}^M \alpha_{nM} + \widetilde{\alpha}_{-n}^M \widetilde{\alpha}_{nM}\big)\Big). \tag{2.96}$$

For open string, depending on the boundary conditions (2.69) i.e. Neumann (N) and Dirichlet (D), left and right modes are related by $\widetilde{\alpha}_n^M = \pm\alpha_n^M$ i.e. sin / cos waves. We list below the possible mode expansions and their Hamiltonians:

(NN) $(\partial_\sigma X^M(\tau,\sigma=0) = \partial_\sigma X^M(\tau,\sigma=l) = 0)$

$$X^M(\tau,\sigma) = x^M + \frac{2\pi\alpha'}{l}p^M\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^M}{n}\,\mathrm{e}^{-i\frac{\pi}{l}n\tau}\cos\Big(\frac{n\pi\sigma}{l}\Big), \tag{2.97a}$$

$$H = \frac{\pi}{2l}\Big(2\alpha' p^2 + \sum_{n\neq 0}\alpha_{-n}^M \alpha_{Mn}\Big). \tag{2.97b}$$

(DD) $(X^M(\tau,\sigma=0) = x_0^M$ and $X^M(\tau,\sigma=l) = x_1^M)$

$$X^M(\tau,\sigma) = x_0^M + \frac{x_1^M - x_0^M}{l}\sigma + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^M}{n}\,\mathrm{e}^{-i\frac{\pi}{l}n\tau}\sin\Big(\frac{n\pi\sigma}{l}\Big), \tag{2.98a}$$

$$H = \frac{1}{4\pi\alpha' l}\big(x_1^M - x_0^M\big)^2 + \frac{\pi}{2l}\sum_{n\neq 0}\alpha_{-n}^M\alpha_{Mn}. \tag{2.98b}$$

(ND) $(\partial_\sigma X^M(\tau,\sigma=0) = 0$ and $X^M(\tau,\sigma=l) = x_0^M)$

$$X^M(\tau,\sigma) = x_0^M + i\sqrt{2\alpha'}\sum_{n\in Z+\frac{1}{2}}\frac{\alpha_n^M}{n}\mathrm{e}^{-i\frac{\pi}{l}n\tau}\cos\Big(\frac{n\pi\sigma}{l}\Big), \tag{2.99a}$$

$$H = \frac{\pi}{2l}\sum_{n\in Z+1/2}\alpha_{-n}^M\alpha_{Mn}. \tag{2.99b}$$

The momentum operator appears only in the (NN) directions.

For canonical quantization, we replace the Poisson bracket $\{X^M(\tau,\sigma),\Pi^N(\tau,\sigma')\}_{\mathrm{PB}} = \eta^{MN}\delta(\sigma-\sigma')$ by the commutator. In terms of the mode expansions, we find the following commutation relations:

$$[x^M,\ p^N] = i\eta^{MN}, \quad [\alpha_m^M,\ \alpha_n^N] = [\widetilde{\alpha}_m^M,\ \widetilde{\alpha}_n^N] = m\delta_{m+n}\eta^{MN}, \quad [\alpha_m^M,\ \widetilde{\alpha}_n^N] = 0. \tag{2.100}$$

By using the modes (2.100), we could construct Fock space, defining the ground state $|0;k\rangle$ as $\hat{p}^M|0;k\rangle = k^M|0;k\rangle$ and $\alpha_m^M|0;k\rangle = \widetilde{\alpha}_m^M|0;k\rangle = 0$ for $m > 0$. In order to obtain physical Hilbert space, we need to impose the physical condition governed by (2.93). This condition indeed removes such as a negative norm state following from $[a_m^0,\ a_m^{0\,\dagger}] = -1$ with $(a_m^M,\ a_m^{M\dagger}) \equiv (\alpha_m^M/\sqrt{m},\ \alpha_{-m}^M/\sqrt{m})$. We first define the Virasoro generators via the Fourier transformation of the energy momentum tensors:

$$L_m \equiv -\frac{l}{4\pi^2}\int_0^l \mathrm{d}\sigma\, T_{--}\mathrm{e}^{-i\frac{2\pi}{l}m\sigma} = \frac{1}{2}\sum_n \alpha_{m-n}^M\alpha_{nM}, \quad \widetilde{L}_m \equiv -\frac{l}{4\pi^2}\int_0^l \mathrm{d}\sigma\, T_{++}\mathrm{e}^{i\frac{2\pi}{l}m\sigma} = \frac{1}{2}\sum_n \widetilde{\alpha}_{m-n}^M\widetilde{\alpha}_{nM}, \tag{2.101}$$

where we denote $\alpha_0^M = \widetilde{\alpha}_0^M = \sqrt{\alpha'/2}\,p^M$ for closed string and $\alpha_0^M = \sqrt{2\alpha'}p^M$ for open string. The Virasoro generators L_0 and $\widetilde{L}_0$ are related to the Hamiltonian

$$H = \frac{2\pi}{l}(L_0 + \widetilde{L}_0), \quad \text{(for closed string)} \qquad H = \frac{\pi}{l}L_0, \quad \text{(for open string)}. \tag{2.102}$$

As usual in the canonical quantization, one should take the normal ordering of operators where all lowering operators are placed to the right. In order to construct the Virasoro generators (2.101) as quantum operators, L_0 and $\widetilde{L}_0$ should be treated with some care. We define L_0 and $\widetilde{L}_0$ with a common constant a which will be determined later:

$$L_0 \longrightarrow L_0 - a = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}^M \alpha_{nM} - a, \quad \widetilde{L}_0 \longrightarrow \widetilde{L}_0 - a = \frac{1}{2}\widetilde{\alpha}_0^2 + \sum_{n=1}^{\infty} \widetilde{\alpha}_{-n}^M \widetilde{\alpha}_{nM} - a. \tag{2.103}$$

Then, we can define the Hilbert space which satisfies the condition

$$(L_m - a\delta_m)|\text{phys}\rangle = (\widetilde{L}_m - a\delta_m)|\text{phys}\rangle = 0, \quad m \geq 0, \quad \text{and} \quad (L_0 - \widetilde{L}_0)|\text{phys}\rangle = 0. \tag{2.104}$$

Due to the hermiticity of the Virasoro generators $L_m^\dagger = L_{-m}$, we only need to impose the physical state condition (2.104) for $m \geq 0$. As we will see, the conditions (2.104) for L_0 and $\widetilde{L}_0$ yield the mass relation. The Virasoro operators $L_{-m}(m > 0)$ act on physical states and create the highest representation of the Virasoro algebra. In 26D, these states can be identified with the gauge degrees of freedom (null states).

Let us discuss the normal ordering constant a. In practical computations, the normal ordering constant is related to the following relation,

$$\frac{1}{2}\sum_n \alpha_{-n}\alpha_n = \frac{1}{2}\sum_n : \alpha_{-n}\alpha_n : + \frac{1}{2}\sum_{n=1}^{\infty} n, \quad \text{with} \quad [\alpha_m,\ \alpha_n] = m\delta_{m+n}. \tag{2.105}$$

Indeed the constant a for closed string is formally given by

$$a_{\text{closed}} = (D-2) \times \Big(- \frac{1}{2}\sum_{n=1}^{\infty} n \Big). \tag{2.106}$$

It is clear in the light-cone gauge that the coefficient $(D-2)$ can be interpreted as the number of physical transverse modes of a string in D dimensions. As we have used before, we could here also adopt the zeta function regularization; $\sum_{n=1}^{\infty} n = \zeta(-1) = -1/12$.

We here consider open strings in D dimensions which consist of n (ND) directions and $(D-n)$ (NN) and (DD) directions. Regarding the normal ordering, (NN) and (DD) directions have the same structure as (2.105) because of the "integer" modes, i.e. for each direction,

$$a_{\text{NN/DD}} = \frac{1}{24}. \tag{2.107}$$

However, for (ND) directions (2.99a), those contain half-integer (more correctly half-odd integer) modes. The same discussion that led to (2.107) gives the constant a for one (ND) string as[10]

$$a_{\text{ND}} = -\frac{1}{2}\sum_{n=1}^{\infty}\Big(n + \frac{1}{2}\Big) = -\frac{1}{48}. \tag{2.108}$$

Therefore, the open string has the normal ordering constant

$$a_{\text{open}} = \frac{1}{24}\big(\#(\text{NN}) + \#(\text{DD})\big) - \frac{1}{48}\big(\#(\text{ND})\big) = \frac{D-2}{24} - \frac{n}{16}. \tag{2.109}$$

The mass M^2 of closed and open string excitations with n (ND) directions can be estimated through the relation (2.104)

$$M_{\text{closed}}^2 \equiv -p^2 = \frac{2}{\alpha'}\big(N_{\text{closed}} + \widetilde{N}_{\text{closed}} - 2a_{\text{closed}}\big), \quad \text{with} \quad N_{\text{closed}} = \widetilde{N}_{\text{closed}}, \tag{2.110a}$$

$$M_{\text{open}}^2 \equiv -p^2 = \frac{1}{\alpha'}\big(N_{\text{open}} - a_{\text{open}}\big) + \frac{(x_0 - x_1)^2}{(2\pi\alpha')^2}. \tag{2.110b}$$

[10]We use $\zeta(s,q) = \sum_{n=1}^{\infty}(n+q)^{-s}$ with analytic continuation $\zeta(-1,q) = -(6q^2 - 6q + 1)/12$.

The number operators N are given by

$$N_{\text{closed}} = \sum_{n=1}^{\infty} \alpha^M_{-n}\alpha_{Mn}, \quad \widetilde{N}_{\text{closed}} = \sum_{n=1}^{\infty} \widetilde{\alpha}^M_{-n}\widetilde{\alpha}_{Mn}, \quad N_{\text{open}} = \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha_{in} + \sum_{n\in Z+\frac{1}{2}>0} \alpha^a_{-n}\alpha_{an}, \tag{2.111}$$

where i and a run through (NN), (DD) directions and (ND) directions, respectively. In 26D, the normal ordering constants may be $a_{\text{closed}} = 1$ and $a_{\text{open}} = 1 - n/16$. We can observe the tension of (ND) directions contribute to the open string mass spectrum (2.110b).

Now, we could construct the Hilbert space. For closed strings, the mass-shell relation (2.110a) tells that the ground state $|0; k\rangle$ is a tachyonic scalar. We could construct the following massless states as the first excited states,

$$\begin{aligned} \alpha^{\{M}_{-1}\widetilde{\alpha}^{N\}}_{-1} - \tfrac{1}{D}\eta^{MN}(\alpha_{-1}\cdot\widetilde{\alpha}_{-1})|0;k\rangle &: \quad \text{symmetric tensor } G_{MN}(x) \\ \alpha^{[M}_{-1}\widetilde{\alpha}^{N]}_{-1}|0;k\rangle &: \quad \text{antisymmetric tensor } B_{MN}(x) \\ \tfrac{1}{D}(\alpha_{-1}\cdot\widetilde{\alpha}_{-1})|0;k\rangle &: \quad \text{scalar } \Phi(x) \end{aligned}$$

Virasoro constraints produce the condition (2.75) for gravitons.

Let us move on to open strings. For simplicity, we here discuss the case with no (ND) directions, i.e. $a_{\text{open}} = 1$. Moreover, we consider a particular open string in Fig.3, i.e. $x^a_1 = x^a_0$. As in the closed strings, the ground state is tachyonic. The first excited states are given by

$$\alpha^m_{-1}|0;k\rangle \quad \text{and} \quad \alpha^a_{-1}|0;k\rangle\text{:} \quad \text{vector } A_m(\sigma) \quad \text{and} \quad \text{scalar } \phi_a(\sigma)$$

where m and a denote the (NN) and (DD) directions, respectively. From (2.110b), these are massless states. The momentum k depends only on the direction for Neumann boundary conditions. The scalar fields $\phi_a(\sigma)$ correspond to the position of D-brane. More about D-branes will be given in later.

Both for closed and open strings, in addition to these massless modes, infinitely many massive states are measured by the inverse of string length $1/\sqrt{\alpha'}$. Sending $\alpha' \to 0$, the massive modes may be decoupled.

2.4.5 Closed string one-loop

Let us briefly discuss closed string vacuum amplitudes. As we discussed, in general, string Feynman diagram can be classified by the genus expansions. The first nontrivial contribution to the partition function is string one-loop, i.e. torus. One could calculate this by essentially the same way done in the case of particles in closed path (2.53).

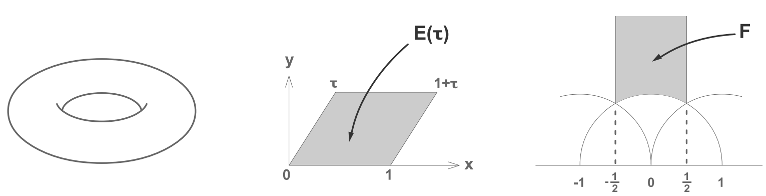

Figure 6: Torus, complex structure, and fundamental domain

Regarding a torus world-sheet, we first specify physical deformation of the torus which corresponds to the length of the world-line l for particle trajectory i.e. S^1 for closed path. Since a torus has two non-contractible cycles $S^1 \times S^1$, there may exist two parameters which characterize the torus. This can be represented by a complex parameter $\tau = \tau_1 + i\tau_2$ in the complex plane called moduli parameter, which remains after suitable gauge fixing for the diffeomorphism and Weyl rescaling and should be integrated in the path-integral. A torus can be constructed by a parallelogram $E(\tau)$ with

identifications of two sides. There exists the "large" coordinate transformations $\tau' = (a\tau + b)/(c\tau + d)$ with $ad - bc = 1$, $(a, b, c, d \in Z)$ generated by $\tau' = \tau + 1$ and $\tau' = -1/\tau$ which could not be absorbed by diffeomorphism/Weyl rescaling. This deformation is called as moduli. It turns out that topologically inequivalent torus with τ can be defined in so-called the fundamental domain F which is usually taken to be,

$$F = \left\{ \tau_2 > 0, \quad -\frac{1}{2} < \tau_1 \leq \frac{1}{2}, \quad |\tau| \geq 1 \right\}. \tag{2.112}$$

The path-integration should be performed over the fundamental domain F. By doing the diffeomorphism and the Weyl rescaling on the world sheet, we could bring the world-sheet metric to be constant which is

$$\gamma_{ab} = \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix} \quad \Longleftarrow \quad \mathrm{d}s^2 = |\mathrm{d}z|^2 = \gamma_{ab}\mathrm{d}x^a\mathrm{d}x^b, \quad \text{with} \quad z = 1 \cdot x^1 + \tau x^2,\ 0 \leq x^1,\ x^2 \leq 1\,, \tag{2.113}$$

where z specifies the point on the torus with coordinates (x^1, x^2). This corresponds to fixing the gauge as $e = l$ in the particle case. Using the preparations above, one could evaluate the path-integral in the essentially same way done in the case of particles. Choosing suitable "eigenfunctions" with periodic boundary conditions, we could obtain

$$Z_{\text{torus}} = \int \frac{\mathcal{D}X\mathcal{D}\gamma}{V_{\text{gauge}}}\, \mathrm{e}^{-\widetilde{S}_1[X,\gamma]} \propto \int_F \frac{\mathrm{d}^2\tau}{4\tau_2}(4\pi^2\alpha'\tau_2)^{-13}|\eta(\tau)|^{-48}, \tag{2.114}$$

where $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n)$ with $q = \mathrm{e}^{i2\pi\tau}$ is the Dedekind's eta function.

Since the string partition function should be the same for equivalent tori, it should be invariant under the modular transformations. One can indeed check this with the result in (2.114). The modular invariance is an important concept in string theory. We could require this to construct consistent string theories. One of the examples is the GSO projection in Type II superstring theories.

It is interesting to compare the results between the case of particle (2.53) and string (2.114). The UV divergence which has been observed in the case of particle with $l \to 0$ does not exist in the string case where $\tau_2 \sim l$ is bounded below in the fundamental domain. The geometrical restriction of torus topology brings natural cut-off in string theory.

On the other hand, for IR region i.e. large $\tau_2 \sim l$, let us compare the result for the one-loop amplitude (2.114) after performing τ_1 integration and that for the particle (2.53) in $D = 26$,

$$Z_{\text{torus}} \sim \int \frac{\mathrm{d}l}{l}\frac{\mathrm{e}^{2l}}{l^{13}} \quad \Longleftrightarrow \quad \int \frac{\mathrm{d}l}{l}\frac{\mathrm{e}^{-\frac{1}{2}lm^2}}{l^{26/2}} = Z_{S^1}\,. \tag{2.115}$$

This indicates that there exists the tachyon with mass $m^2 = -4$. This is the lightest mode of bosonic string which could be first observed in this low energy limit. This IR divergence could be cured by introducing supersymmetry, and thus we could observe the finiteness of string theories. It is believed that the finiteness exists in the higher loops.

2.5 Superstring

By definition, in bosonic string theory so far discussed, there are no fermionic fields in spacetime/world sheet. Moreover in the bosonic theory, the ground state is tachyonic[11]. These are resolved if the supersymmetry is introduced. As a result, requiring the tachyon and the anomaly free, five superstring theories (Type I, Type IIA, Type IIB, Heterotic with gauge groups $E_8 \times E_8$ and $SO(32)$) can be

[11] In field theories, the presence of tachyon means that one chooses the wrong vacuum. This may be also the case in string theories [25].

consistently formulated in 10D target spacetime [26]. We only consider Type II theories in this paper. These possess $\mathcal{N} = 2$ supersymmetry: Type IIA has (1,1) supersymmetry with non chiral spinors while Type IIB is the chiral theory with (2, 0) supersymmetry. Type II superstrings are for closed string theories. However, these may provide a seat for D-brane, so that open string could participate and play an interesting role.

2.5.1 Quantization of superstring in flat background

Let us enlarge the world-sheet symmetry to 2D local supersymmetry. First of all, we need to introduce superpartners of the bosonic fields $(X^M(\sigma), \gamma_{mn}(\sigma))$ in the Polyakov action (2.90), i.e. D fermionic "matter" $\psi^M(\sigma)$ and gravitino $\chi_m(\sigma)$. This approach leads to Neveu-Schwarz-Ramond superstring [27] [12]. The resultant supersymmetric action can be found in [29]. By using diffeomorphism, Weyl, and their fermionic counterparts, one could take the superconformal gauge i.e. $\gamma_{mn}(\sigma) = \eta_{mn}$ and $\chi_m(\sigma) = 0$. The bosonic part becomes the same as (2.91), while the fermionic part may be given by

$$S_{\rm F} = -\frac{1}{2\pi}\int \mathrm{d}^2\sigma\, i\big(\bar{\psi}^M \rho^m \partial_m \psi_M\big) = \frac{i}{\pi}\int \mathrm{d}^2\sigma\, \big(\psi_+^M \partial_- \psi_{+M} + \psi_-^M \partial_+ \psi_{-M}\big), \tag{2.116}$$

where we have taken 2D gamma matrix as $\rho^0 = i\sigma^2$ and $\rho^1 = \sigma^1$. The fermionic matters $\psi^M = (\psi_+^M\ \psi_-^M)^{\rm T}$ are Majorana-Weyl fermion whose two components are real Grassmann fields in this representation. The field variation of (2.116) gives

$$\delta S_{\rm F} = \frac{2i}{\pi}\int \mathrm{d}^2\sigma\big(\delta\psi_+^M \partial_- \psi_{+M} + \delta\psi_-^M \partial_+ \psi_{-M}\big) + \frac{i}{2\pi}\int \mathrm{d}\tau\big(\delta\psi_+\psi_+ - \delta\psi_-\psi_-\big)\big|_{\sigma=0}^{\sigma=l}. \tag{2.117}$$

The first part gives equations of motion

$$0 = \partial_+\psi_- = \partial_-\psi_+, \tag{2.118}$$

while the second part reads boundary terms which should vanish with appropriate boundary conditions. For closed strings, these should be done by imposing the periodic/antiperiodic boundary conditions $\psi_\pm^M(l) = \eta\psi_\pm^M(0)$ with $\eta^2 = 1$. For open strings, we need to impose conditions between left/right movers at the boundaries: $\psi_\pm^M(0) = \psi_\mp^M(0)$ and $\psi_\pm^M(l) = \eta\psi_\mp^M(l)$.

Let us next consider residual supersymmetries after taking the superconformal gauge. Under transformations

$$\delta X^M = 2i\alpha'\bar{\epsilon}\psi^M = 2i\alpha'(\epsilon_+\psi_-^M - \epsilon_-\psi_+^M), \tag{2.119a}$$
$$\delta\psi^M = (\rho^m\epsilon)\partial_m X^M, \qquad \text{i.e.} \qquad \delta\psi_\pm^M = \pm 2\epsilon_\mp\partial_\pm X^M, \tag{2.119b}$$

the total action (2.91) and (2.116) transforms as

$$\begin{aligned}\delta\big(S_B + S_F\big) = \frac{4i}{\pi}&\int \mathrm{d}^2\sigma\big((\partial_+\epsilon_+)\psi_-^M\partial_- X_M - (\partial_-\epsilon_-)\psi_+^M\partial_+ X_M\big)\\ &-\frac{i}{\pi}\int \mathrm{d}\tau\big(\epsilon_+\psi_-^M\partial_+ X_M + \epsilon_-\psi_+^M\partial_- X_M\big)\big|_{\sigma=0}^{\sigma=l},\end{aligned} \tag{2.120}$$

where $\epsilon(\sigma) = (\epsilon_+(\sigma), \epsilon_-(\sigma))^{\rm T}$ are infinitesimal supersymmetric parameters. Therefore, residual symmetries are realized by holomorphic functions $\epsilon_\pm(\sigma) = \epsilon_\pm(\sigma^\mp)$ with suitable boundary conditions which

[12] The Polyakov action is invariant under the global Poincaré transformation. One can enlarge this symmetries to global invariance of the super Poincaré group. This leads to an another formulation of superstring so-called Green-Schwarz superstring [28].

make boundary terms vanish. Together with the bosonic part, residual symmetry becomes superconformal symmetry.

Now let us discuss the mode expansions taking the boundary terms in (2.117) and (2.120) into account. For closed fermionic strings, we obtain the following solutions,

$$\psi_{\pm}^{M}(\tau,\sigma)=\sqrt{\frac{2\pi}{l}}\sum_{n\in Z}d_{n}^{M}\,\mathrm{e}^{-i\frac{2\pi}{l}n\sigma_{\pm}}\qquad\text{and}\qquad\psi_{\pm}^{M}(\tau,\sigma)=\sqrt{\frac{2\pi}{l}}\sum_{n\in Z+\frac{1}{2}}b_{n}^{M}\,\mathrm{e}^{-i\frac{2\pi}{l}n\sigma_{\pm}}\tag{2.121}$$

The first one which satisfies the periodic boundary condition is called Ramond sector, while the second one that satisfies anti-periodic boundary condition is called Neveu-Schwarz sector. Therefore, in closed strings which contain independent left/right movers, there exist four sectors i.e. Ramond-Ramond, (Neveu-Schwarz)-(Neveu-Schwarz), (Neveu-Schwarz)-Ramond and Ramond-(Neveu-Schwarz) sectors. The consistency at string loop level requires that all four sectors should be included.

For open strings, we have options for choosing the Neumann/Dirichlet boundary conditions for bosonic strings (2.97a), (2.98a) and (2.99a). Since supersymmetries relate bosonic/fermionic strings, we could also take various directions of fermionic strings such as (NN), (ND), and (DD) directions. Defining Ramond and Neveu-Schwarz sector in (NN) directions as

(NN)

$$\text{(R)}\qquad\psi_{\pm}^{M}(\tau,\sigma)=\sqrt{\frac{\pi}{l}}\sum_{n\in Z}d_{n}^{M}\,\mathrm{e}^{-i\frac{\pi}{l}n\sigma_{\pm}}\qquad\text{(NS)}\quad\psi_{\pm}^{M}(\tau,\sigma)=\sqrt{\frac{\pi}{l}}\sum_{n\in Z+\frac{1}{2}}b_{n}^{M}\,\mathrm{e}^{-i\frac{\pi}{l}n\sigma_{\pm}}\tag{2.122a}$$

which could fix boundary conditions of the supersymmetric parameters, we can sequentially find mode expansions for each sectors in another directions:

(DD)

$$\text{(R)}\qquad\psi_{\pm}^{M}(\tau,\sigma)=\pm\sqrt{\frac{\pi}{l}}\sum_{n\in Z}d_{n}^{M}\,\mathrm{e}^{-i\frac{\pi}{l}n\sigma_{\pm}}\qquad\text{(NS)}\quad\psi_{\pm}^{M}(\tau,\sigma)=\pm\sqrt{\frac{\pi}{l}}\sum_{n\in Z+\frac{1}{2}}b_{n}^{M}\,\mathrm{e}^{-i\frac{\pi}{l}n\sigma_{\pm}}\tag{2.122b}$$

(ND)

$$\text{(R)}\qquad\psi_{\pm}^{M}(\tau,\sigma)=\sqrt{\frac{\pi}{l}}\sum_{n\in Z+\frac{1}{2}}d_{n}^{M}\,\mathrm{e}^{-i\frac{\pi}{l}n\sigma_{\pm}}\qquad\text{(NS)}\quad\psi_{\pm}^{M}(\tau,\sigma)=\sqrt{\frac{\pi}{l}}\sum_{n\in Z}b_{n}^{M}\,\mathrm{e}^{-i\frac{\pi}{l}n\sigma_{\pm}}\tag{2.122c}$$

It should be noted that in the (ND) directions the Ramond and the Neveu-Schwarz sectors turn to have half-integer and integer mode numbers, respectively. As in the closed string, Hilbert space should contain both R and NS sectors.

We now quantize the fermionic string canonically, given the anti-commutation relation:

$$\{d_{m}^{M},\ d_{n}^{N}\}=\{b_{m}^{M},\ b_{n}^{N}\}=\delta_{m+n}\eta^{MN}.\tag{2.123}$$

The ground state of NS sector is defined by $\alpha_{m}^{M}|0\rangle_{\mathrm{NS}}=b_{n}^{M}|0\rangle_{\mathrm{NS}}=0$ $(m>0,n>0)$, and similar conditions should be satisfied for $\widetilde{\alpha}_{m}^{M}$ and $\widetilde{b}_{n}^{M}$ in closed string. As the definition above, the ground state $|0\rangle_{\mathrm{NS}}$ is unique and describes the spacetime scalar. We could build up the Fock space on the ground state by multiplying the oscillator modes α_{-m}^{M} and b_{-r}^{M} $(m>0,r>0)$.

For R sector, non-zero modes d_{n}^{M} $(n>0)$ play a similar role, $\alpha_{m}^{M}|0\rangle_{\mathrm{R}}=d_{n}^{M}|0\rangle_{\mathrm{R}}=0$. However, we observe zero-mode parts of the anti-commutation relations in (2.123) follow the Clifford algebra with replacements $d_{0}^{M}=\Gamma^{M}/\sqrt{2}$,

$$\{d_{0}^{M},\ d_{0}^{N}\}=\eta^{MN}.\tag{2.124}$$

This indicates the Ramond ground states represent the Clifford algebra in D-dimensional spacetime, i.e. spacetime spinor. The world-sheet supersymmetry has generated spacetime fermion.

As we did in the bosonic string, we would like to construct mass relations. In order to fix the normal ordering constant, we use the zeta function regularization. The result is given by

$$a_{\text{integer mode}} = -\frac{1}{24}, \qquad a_{\text{half-integer mode}} = +\frac{1}{48}, \tag{2.125}$$

which would be compared with (2.107) and (2.108). By using these constants, we could estimate the mass relation

$$M^2_{\text{open}} = \frac{1}{\alpha'}\big(N_X + N_\psi^{\text{R}}\big) + \frac{(x_0 - x_1)^2}{(2\pi\alpha')^2}, \qquad \text{(R)} \tag{2.126a}$$

$$M^2_{\text{open}} = \frac{1}{\alpha'}\big(N_X + N_\psi^{\text{NS}}\big) + \frac{(x_0 - x_1)^2}{(2\pi\alpha')^2} - \frac{1}{\alpha'}\Big(\frac{1}{2} - \frac{n}{8}\Big), \qquad \text{(NS)}\,, \tag{2.126b}$$

where the newly added number operator of the fermionic part is given by $N_\psi^{\text{R}} = \sum_{n>0} n d_{-n}^M d_{nM}$ and $N_\psi^{\text{NS}} = \sum_{n>0} n b_{-n}^M b_{nM}$. M runs through (NN), (DD), (ND) directions and n should be understood as integer/half-integer depending on the directions. The mass spectrum does depend on the number of (ND) directions n.

We need to impose so-called GSO projection. We first introduce an operator F which counts the number of oscillators $F_{\text{NS}} = \sum_{r>0} b_{-r} b_r$ and $F_{\text{R}} = \sum_{r>0} d_{-r} d_r$ for each sectors. We now define the operator $G = (-)^{F_{\text{NS}}+1}$ and $G = \Gamma^{11}(-)^{F_{\text{R}}}$ with the 10D chirality operator Γ^{11}. Then GSO projection implies $G|\text{phys}\rangle = |\text{phys}\rangle$. Through this projection, in the NS sector, states with only odd numbers of oscillators could survive in the Hilbert space. This removes tachyonic ground state and keeps the first excited states which correspond to the massless gauge (scalar) fields. In the R sector, we could project states with either even or odd numbers of oscillator excitations depending on the chirality of the spinor ground state. The GSO projection provides spacetime supersymmetry and also ensures the modular invariance. Depending on keeping/removing states, we obtain different theories. Type IIA/IIB is just in this category.

2.5.2 Supergravity

Superstring contains a spacetime supersymmetry and also includes gravity. Therefore, the low energy effective action could be that of supergravity. Bosonic contents of the supergravity multiplets are a graviton and its family ($G_{MN}(x)$, $B_{MN}(x)$, $\Phi(x)$) as in the common sector referred to as NSNS sector, and p-form gauge fields $C_p(x)$ with $p = 1, 3$ for type IIA and with $p = 0, 2, 4$ for type IIB in RR sector. Other sectors in closed strings i.e. RNS and NSR provide superpartners of those fields. In this paper we only display these bosonic contents in the supergravity theories. Note that in RR sectors in Type II string theories, physical states are identified not with gauge potentials but with their field strength. In other words, there exist no states with RR charges in string perturbation theories. As we will discuss, RR charge can appear through D-branes in nonperturbative string theory.

The common NSNS sector of IIA/IIB supergravity action is the same with (2.87) with

$$D = D_{\text{c}} = 10. \tag{2.127}$$

The actions for p-form gauge fields in RR sector are given by

$$S_{\text{RR}}^{\text{IIA}} = -\frac{1}{4\kappa_0^2}\int_{\mathcal{M}} \mathrm{d}^{10}x\sqrt{-G}\Big(|F_2|^2 + |\widetilde{F}_4|^2\Big) - \frac{1}{4\kappa_0^2}\int_{\mathcal{M}} B_2 \wedge F_4 \wedge F_4, \tag{2.128a}$$

$$S_{RR}^{\text{IIB}} = -\frac{1}{4\kappa_0^2}\int_{\mathcal{M}} \mathrm{d}^{10}x\sqrt{-G}\Big(|F_1|^2 + |\widetilde{F}_3|^2 + \frac{1}{2}|\widetilde{F}_5|^2\Big) - \frac{1}{4\kappa_0^2}\int_{\mathcal{M}} C_4 \wedge H_3 \wedge F_3, \tag{2.128b}$$

with

$$\widetilde{F}_4 = F_4 - C_1 \wedge H_3, \qquad \widetilde{F}_3 = F_3 - C_0 \wedge H_3, \qquad \widetilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3 \, .$$

Here we define the field strength of so-called RR p-form gauge fields as $F_{p+1}(x) = \mathrm{d}C_p(x)$ and denote $H_3(x) = \mathrm{d}B_2(x)$ for the Kalb-Ramond two-form field. We need to impose the self-dual condition $*\widetilde{F}_5(x) = F_5(x)$ at the level of the equations of motion for IIB. The massless closed superstring spectrum does not only contain the graviton and its family but also RR p-forms. Apart from the Chern-Simons terms which are the second parts of the actions (2.128a) and (2.128b), schematically, we write the low energy effective action for type II superstring theories as

$$S_{\mathrm{NSNS-RR}p} = \frac{1}{2\kappa_0^2 g_{\mathrm{s}}^2} \int_{\mathcal{M}} \mathrm{d}^{10}x \sqrt{-G} \Big\{ \mathrm{e}^{-2\widetilde{\Phi}} \Big(R + 4(\partial_M \widetilde{\Phi})^2 - \frac{1}{12} H_{MNL} H^{MNL} \Big) - \frac{g_{\mathrm{s}}^2}{2} |F_{p+2}|^2 \Big\}, \qquad (2.129)$$

where we have divided $\Phi(x) = \Phi_0 + \widetilde{\Phi}(x)$ as before. The action (2.129) is defined in the string frame which is constructed from the string perturbation theory. In order to obtain the canonical form of the Einstein-Hilbert action we could rescale the string frame metric $G_{MN}(x)$ to that of the Einstein frame $G^{\mathrm{E}}_{MN}(x)$ through the Weyl rescaling,

$$G^{\mathrm{E}}_{MN}(x) = \mathrm{e}^{-\widetilde{\Phi}(x)/2} G_{MN}(x), \qquad (2.130)$$

so that

$$S_{\mathrm{NSNS-RR}p} = \frac{1}{2\kappa_0^2 g_{\mathrm{s}}^2} \int_{\mathcal{M}} \mathrm{d}^{10}x \sqrt{-G^{\mathrm{E}}} \Big\{ R(G^{\mathrm{E}}) - \frac{1}{2} (\partial_M \widetilde{\Phi})^2 - \frac{1}{12} \mathrm{e}^{-\widetilde{\Phi}} H_{MNL} H^{MNL} - \frac{g_{\mathrm{s}}^2}{2} \, \mathrm{e}^{\frac{3-p}{2}\widetilde{\Phi}} |F_{p+2}|^2 \Big\}. \qquad (2.131)$$

Here the kinetic term of the dilaton field also becomes its canonical form. The canonical form for Einstein-Hilbert action is used in this paper to estimate the masses of the objects. By using this form, we define the 10D Newton constant $G^{(10)}_{\mathrm{N}}$ as

$$16\pi G^{(10)}_{\mathrm{N}} = 2\kappa_0^2 g_{\mathrm{s}}^2 \equiv 2\kappa^2. \qquad (2.132)$$

In the dimensional analysis, the gravitational coupling constant can be estimated in terms of the string length $\sqrt{\alpha'}$,

$$\kappa \propto \alpha'^2. \qquad (2.133)$$

2.5.3 RR p-brane solutions

In supergravity theories, there exist various classes of exact solutions for equations of motion derived from the supergravity action [30]. Let us consider p-brane solitonic solutions which couple to $(p+1)$-form gauge fields. We are interested in the classical solutions which contain minimal subset of the supergravity field contents, i.e. the graviton, the dilaton, and $(p+1)$-from gauge fields. Setting the Kalb-Ramond field $B_{MN}(x) = 0$, we consider the following truncated action:

$$\widetilde{S}_{\mathrm{NSNS-RR}p} = \frac{1}{2\kappa_0^2 g_{\mathrm{s}}^2} \int_{\mathcal{M}} \mathrm{d}^{10}x \sqrt{-G} \Big\{ \mathrm{e}^{-2\widetilde{\Phi}} \Big(R + 4(\partial_M \widetilde{\Phi})^2 \Big) - \frac{g_{\mathrm{s}}^2}{2} |F_{p+2}|^2 \Big\}. \qquad (2.134)$$

We would like to find a solution like a static charged point particle. The "point" would be replaced to the p-dimensional flat object, and the transverse directions to the object might have spherical symmetry where the object looks charged "point". In addition, in order to isolate the extended object, we need to

impose asymptotically flat condition to the transverse directions. With these ansatz, the solution has the following form in $p < 7$ in the string frame [31];

$$\begin{aligned}
\mathrm{d}s^2 &= \frac{1}{\sqrt{H_p(r)}}\Big(- f(r)\mathrm{d}t^2 + \sum_{i=1}^{p} \mathrm{d}x^{i2}\Big) + \sqrt{H_p(r)}\Big(\frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega_{8-p}^2\Big), \\
\mathrm{e}^{\widetilde{\Phi}} &= H_p^{(3-p)/4}(r), \\
F_{p+2} &= \frac{1}{g_\mathrm{s}}\sqrt{1 + \frac{r_\mathrm{H}^{7-p}}{L^{7-p}}}\ \mathrm{d}t \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^p \wedge \mathrm{d}(H_p^{-1}(r)), \quad \Big(\text{or} \quad F_{8-p} = *F_{p+2}\Big),
\end{aligned} \tag{2.135}$$

with

$$H_p(r) = 1 + \frac{L^{7-p}}{r^{7-p}}, \qquad f(r) = 1 - \frac{r_\mathrm{H}^{7-p}}{r^{7-p}}. \tag{2.136}$$

We have parameterized the $(p+1)$-dimensional world volume coordinates as $(t, x^1, \cdots, x^p)$ and the transverse $(9-p)$-dimensional space by the polar coordinate system with radial coordinate $r\Big(= \sqrt{(x^{p+1})^2 + \cdots + (x^9)^2}\ \Big)$. The functions $H_p(r)$ and $f(r)$ are harmonic functions of the transverse coordinates, meaning $0 = (\partial_{p+1}^2 + \cdots + \partial_9^2)H_p(r)$ and the same for $f(r)$. They are normalized such that the metric becomes asymptotically flat in the spatial infinity. In the case of $p = 3$, we need to add the Hodge dual of the expression F_5 in (2.135) to satisfy the self-dual relation $*F_5 = F_5$.

Two integration constants L and r_H could be related with physical parameters i.e. charge and mass. Using the expression (2.63) and taking the normalization of the action (2.134) into account, we can define charge Q_p in the covariant way through the Gauss' law which is an integration of flux over the transverse sphere S^{8-p} (cf. (2.63)),

$$Q_p \equiv \frac{1}{2\kappa_0^2}\int_{S^{8-p}} *F_{p+2} = \frac{V_{S^{8-p}}}{2\kappa_0^2 g_\mathrm{s}}(7-p)L^{\frac{7-p}{2}}\sqrt{L^{7-p} + r_\mathrm{H}^{7-p}}, \tag{2.137}$$

where $V_{S^{8-p}}$ is the volume of unit sphere S^{8-p}. The same result can be obtained in the Einstein frame, although the definition of charge should be modified due to the dilaton factor for the gauge fields in (2.131). It is easy to see that at $r \to \infty$ the electric field behaves like a point charge in the transverse space, which is the generalized Coulomb potential,

$$F_{p+2} = \frac{2\kappa_0^2 g_\mathrm{s} Q_p}{V_{S^{8-p}}}\frac{1}{r^{8-p}}\ \big(\mathrm{d}t \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^p \wedge \mathrm{d}r\big) + \cdots .$$

Since the solution is asymptotically flat in the transverse space, we could define a mass M_p. As mentioned before for the point particle, the mass could be read off from the deviation from the flat metric in the asymptotic region (2.22). In order to proceed, we need to take two things into account. First, the procedure only works for the action (2.2) i.e. in the canonical Einstein frame. Second, our extended object has infinite volume in its longitudinal directions. We first regularize this volume and rewrite the action into the canonical Einstein-Hilbert action for point particle source in the $(1+(9-p))$-dimensional transverse spacetime,

$$\widetilde{S}_{\mathrm{NSNS-RR}p} = \frac{V_p}{2\kappa_0^2 g_\mathrm{s}^2}\int \mathrm{d}^{10-p}x\sqrt{-\hat{G}}\Big(R(\hat{G}) + \cdots\Big),$$

where V_p is the p-dimensional spatial volume of the p-brane and we have eliminated the dilaton factor by using Weyl rescaling formula. The resultant metric is

$$\hat{G}_{tt} = -1 + \left(\frac{7-p}{8-p}L^{7-p} + r_\mathrm{H}^{7-p}\right)\frac{1}{r^{7-p}} + \cdots . \tag{2.138}$$

Comparing with (2.22), we find

$$\frac{M_p}{V_p} = \frac{V_{S^{8-p}}}{2\kappa_0^2 g_s^2}\Big((7-p)L^{7-p} + (8-p)r_{\rm H}^{7-p}\Big). \tag{2.139}$$

This M_p/V_p can be interpreted as the tension which is a natural analog of the mass of the p-brane. The Schwarzschild radius $r_{\rm S}$ can be easily estimated through (2.138),

$$r_{\rm S}^{7-p} = \frac{7-p}{8-p}L^{7-p} + r_{\rm H}^{7-p}. \tag{2.140}$$

In analogy with the charged black hole solution or Reissner-Nordström black hole solution, the solution (2.135) is known as a "black" p-brane solution. In particular, this has two horizons which are related with $r = 0$ and $r_{\rm H}$. To avoid the naked singularities, physical solutions may be restricted in the parameter region $r_{\rm H} \geq 0$. This is reflected in terms of the tension and charge obtained in (2.139) and (2.137) as

$$g_s \frac{M_p}{V_p} \geq Q_p, \tag{2.141}$$

which is known as the BPS bound. The extremal case $r_{\rm H} = 0$ is called BPS configurations which correspond to the ground state. The BPS state implies that (tension)=(charge). The situation can be understood in terms of supersymmetry. In fact, the configuration given in (2.135) preserves a half of 32 real supersymmetry transformations in type II supergravity theories (1/2 BPS states). In the theories controlled by supersymmetric algebra having the central charges, the relation (2.141) describes the mass of a state and its central charge which is the RR p-brane charge in our case. If the inequality is saturated, the unitary representation of the algebra belongs to a special class i.e. the short representation. The relation for the BPS state is a consequence of the supersymmetry algebra which is protected from quantum corrections. Therefore, this minimum energy states are completely stable.

The case with $p = 3$ is special since singularities totally disappear. The dilaton field is constant. Therefore, the black RR 3-brane solution describes a smooth solitonic object in the type IIB supergravity theory. All the other brane solutions contain singularities.

Let us look at the black RR 3-brane solution in the case of $p = 3$ in (2.135),

$$\mathrm{d}s^2 = \frac{1}{\sqrt{1+\dfrac{L^4}{r^4}}}\left\{-\Big(1-\frac{r_{\rm H}^4}{r^4}\Big)\mathrm{d}t^2 + \sum_{i=1}^{3}\mathrm{d}x^{i2}\right\} + \sqrt{1+\frac{L^4}{r^4}}\left\{\frac{\mathrm{d}r^2}{\Big(1-\dfrac{r_{\rm H}^4}{r^4}\Big)} + r^2\mathrm{d}\Omega_5^2\right\}. \tag{2.142}$$

The outer horizon appears at $r = r_{\rm H}$. We further consider the extremal case $r_{\rm H} = 0$:

$$\mathrm{d}s^2 = \frac{1}{\sqrt{1+\dfrac{L^4}{r^4}}}\left\{-\mathrm{d}t^2 + \sum_{i=1}^{3}\mathrm{d}x^{i2}\right\} + \sqrt{1+\frac{L^4}{r^4}}\left\{\mathrm{d}r^2 + r^2\mathrm{d}\Omega_5^2\right\}. \tag{2.143}$$

We divide the spacetime in the metric (2.143) into two regimes $r > L$ and $r < L$. In the region $r \gg L$ the metric (2.143) describes the 10D flat Minkowski spacetime. In the other limit $r \ll L$ where the harmonic function can be approximately $H_3(r) = 1 + L^4/r^4 \to L^4/r^4$, we obtain the metric

$$\mathrm{d}s^2 = \underbrace{\frac{r^2}{L^2}\left\{-\mathrm{d}t^2 + \sum_{i=1}^{3}\mathrm{d}x^{i2}\right\} + \frac{L^2}{r^2}\,\mathrm{d}r^2}_{\mathrm{AdS}_5} + \underbrace{L^2\mathrm{d}\Omega_5^2}_{S^5}. \tag{2.144}$$

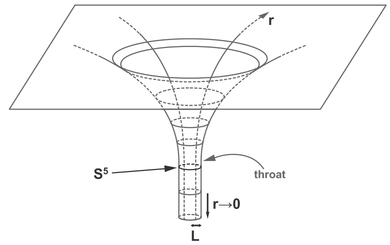

Figure 7: Extremal p-brane

The metric has been divided into two parts, i.e. $\mathrm{AdS}_5 \times S^5$ with the same radius L. We refer to (2.144) as the near horizon geometry where $r = 0$ corresponds to (Killing) horizon. The geometry (2.143) could be visualized as the embedding surface in Fig.7. Two asymptotic regions are separated by an infinitely long "throat" with the constant radius.

The black RR 3-brane solution is a solitonic solution which interpolates two vacua, i.e. flat Minkowski space at infinity and the $\mathrm{AdS}_5 \times S^5$ in the near horizon region. It should be mentioned that these asymptotic vacua preserve all 32 supersymmetries in type IIB supergravity although the full solution breaks half of them [32].

2.6 D-brane

Dp-brane has been introduced through the boundary conditions of open string (2.69) and defined by the $(p+1)$-dimensional hypersurface where the movement of open string endpoints are restricted. This rigid hypersurface turns out to be a dynamical object through the open string ending on it. Moreover, closed string could also interact with D-brane through the open string interactions.

Let us consider the Dp-brane from the viewpoint of the string world sheet. Suppose there is a closed string world sheet. One can consider the interaction with gravity through the vertex operator insertions. Now let us consider the case of the world sheet with boundaries. With the Dirichlet boundary conditions imposed in suitable directions, D-brane could appear at the boundary. Typical world sheet is given by the left side of Fig.8. After appropriate conformal mapping, one can smoothly deform the world sheet

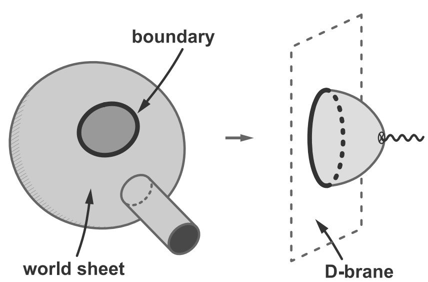

Figure 8: Interactions around D-brane

to the right. This can be interpreted as the interaction between the D-brane and closed string including graviton. One can estimate the strength of the interaction through the world sheet topology. Since the topology of the world sheet is a disc, in terms of the string perturbation (2.78), the weight of the path-integral is given by $1/g_\mathrm{s}$. Since the presence of D-brane can be measured by its tension, we could estimate the tension of Dp-brane as

$$T_p \propto \frac{1}{g_\mathrm{s}}. \tag{2.145}$$

As we have discussed, RR p-branes appear as solitonic solutions in supergravity theory. These have p-form charges, and the extremal solutions are BPS configurations. Polchinski identified this RR p-

brane as the Dp-brane and showed the D-brane is a necessary object for consistency of the interactions between open and closed superstrings [33].

2.6.1 Dp-brane effective action

We could generalize the action (2.88) to Dp-brane effective action by simply replacing[13] $A_M(x) \to (A_m(\sigma),\ \phi_a(\sigma))$, where m is for the world volume coordinate and a is the direction for the Dirichlet boundary condition. The scalar $\phi_a(\sigma)$ corresponds to the collective coordinates of Dp-brane. In this case, we also need to replace the bulk field $G_{MN}(X)$ and $B_{MN}(X)$ by "induced" quantities $g_{mn}(\sigma)$ and $B_{mn}(\sigma)$ defined before,

$$S_{\rm DBI} = -T_p \int_\Sigma \mathrm{d}^{1+p}\sigma\ \mathrm{e}^{-\widetilde{\Phi}} \sqrt{-\det(g_{mn} + B_{mn} + 2\pi\alpha' F_{mn})}. \tag{2.146}$$

Here we fix the Dp-brane tension as[14]

$$T_p = \frac{1}{(2\pi)^p \alpha'^{\frac{p+1}{2}} g_{\rm s}}. \tag{2.147}$$

This is the generalization of Nambu-Goto type action (2.54). The embedding coordinates $X^M(\sigma)$ defined through the induced metric $g_{mn}(\sigma)$ (2.55) represent the shape of the Dp-brane in the target spacetime. We divide the embedding functions to longitudinal and transverse parts to Dp-brane $X^M(\sigma) = (X^m(\sigma), X^a(\sigma) \equiv 2\pi\alpha'\phi^a(\sigma))$, and take the static gauge for the world volume reparameterization invariance $X^m(\sigma) = \sigma^m$.

Let us perturb the DBI action (2.146) around the flat NSNS background in the Einstein frame (2.131),

$$G^{\rm E}_{MN}(x) = \eta_{MN} + \kappa \widehat{h}_{MN}(x), \qquad B_{MN}(x) = 0 + \kappa \widehat{b}_{MN}(x), \qquad \widetilde{\Phi}(x) = 0 + \kappa\widehat{\Phi}(x)\,. \tag{2.148}$$

On this background, we do expand the DBI action for small $\alpha' F_{mn}$. Similar to (2.5), the gravitational coupling constant κ defined in (2.132) appears such that the kinetic terms of the graviton and its family members are correctly normalized. Then, up to the linear order of the perturbation, we find

$$\begin{aligned} S_{\rm DBI} = -T_p(2\pi\alpha')^2 \int_\Sigma \mathrm{d}^{1+p}\sigma \bigg\{ & \frac{1}{(2\pi\alpha')^2} + \frac{\kappa}{(2\pi\alpha')^2}\left(\frac{p-3}{4}\widehat{\Phi} + \frac{1}{2}\widehat{h}_m{}^m\right) + \left(\frac{1}{4}F_{mn}F^{mn} + \frac{1}{2}\partial_m\phi_a\partial^m\phi^a\right) \\ & + \frac{\kappa}{2\pi\alpha'}\left(\widehat{h}_{ma}\partial^m\phi^a + \frac{1}{2}\widehat{b}_{mn}F^{mn}\right) \\ & + \kappa\bigg(\widehat{\Phi}\left(\frac{p-7}{16}F_{mn}F^{mn} + \frac{p-3}{8}\partial_m\phi_a\partial^m\phi^a\right) \\ & \qquad + \frac{1}{2}\widehat{h}_{mn}T^{mn} + \frac{1}{2}\widehat{h}_{ab}\partial_m\phi^a\partial^m\phi^b\bigg) + \cdots \bigg\}, \end{aligned} \tag{2.149}$$

with

$$T_{mn} = -F_{ml}F_n{}^l + \frac{1}{4}\eta_{mn}F_{kl}F^{kl} - \partial_m\phi_a\partial_n\phi^a + \frac{1}{2}\eta_{mn}\partial_l\phi_a\partial^l\phi^a. \tag{2.150}$$

The first part corresponds to the tree level vacuum energy, and the second part gives "source" terms of the dilaton and the graviton that are from massless closed string modes[15]. The third term provides

[13] One of the proper treatments is given via T-duality transformation.

[14] We use the normalization $T_1 \equiv 1/(2\pi\alpha' g_{\rm s}) = T/g_{\rm s}$ for D1-brane. The recursive relation $T_p = 2\pi\sqrt{\alpha'}T_{p+1}$ follows from T-duality.

[15] The source term for the Kalb-Ramond field $B_{mn}(x)$ may not be given by the DBI action but the string world sheet (2.81).

the action for Yang-Mills and scalar fields living in the world volume. Comparing with the canonical normalization of the kinetic term of the vector field, we can relate $p+1$-dimensional Yang-Mills coupling constant to the tension (2.147),

$$g_{\rm YM}^2 = (2\pi)^{p-2}\alpha'^{\frac{p-3}{2}} g_{\rm s}. \tag{2.151}$$

$T_{mn}(x)$ defined in (2.150) can be understood as the energy-momentum tensor in the world volume gauge theory.

We have considered the DBI action (2.146) as the low energy effective action of D-brane which couples to the NSNS background, the graviton, the dilaton, and the Kalb-Ramond field. As we discussed, the Dp-brane naturally couples to the $(p+1)$-form gauge field. The natural candidates for this gauge potentials are RR-gauge potentials in the RR sector. Their low energy couplings are given by the Wess-Zumino term,

$$S_{\rm WZ} = -\mu_p \sum_p \int_{\Sigma_{p+1}} C_{p+1}\wedge {\rm e}^{B+2\pi\alpha' F} = -\mu_p \int_{\Sigma_{p+1}} \Big(C_{p+1} + C_{p-1}\wedge (B+2\pi\alpha' F) + \cdots \Big), \tag{2.152}$$

where the integration should be taken on the Dp-brane world volume. The p-form gauge field $C_p(\sigma)$ and Kalb-Ramond two-form field $B(\sigma)$ have to be understood as the induced quantities of the world volume. The first term is a natural coupling given by (2.59). In addition, there exist nontrivial couplings with lower dimensional RR forms. For instance, for D3-brane, the WZ term (2.152) provides the term

$$-\mu_3(2\pi\alpha')^2 \int C_0 F\wedge F = \mu_3 (2\pi\alpha')^2 \int {\rm d}^4\sigma \sqrt{-g}\Big(\frac{C_0}{4}\epsilon^{mnkl}F_{mn}F_{kl}\Big),$$

where the scalar field $C_0(\sigma)$ corresponds to the axion which may fix the theta angle. We will also use the WZ term to construct baryons in the holographic QCD.

Through the open string fluctuations we obtained the Yang-Mills system in $(1+p)$ dimensions as the low energy effective action of Dp-brane. Instead of a single D-brane, we now consider a stack of N parallel D-branes which are placed on top of each other. There are now N^2 different types of open strings, where each type of string starts from a particular one of the N branes and ends on a particular one[16] . As we have discussed, the endpoint of string is charged under $U(1)$ gauge field. We assume that the low-energy theory describing such a stack of branes has basically the same form with the theory we have discussed, but the fields are promoted to $N\times N$ matrices i.e. $A_m \to (A_m)^i{}_j$ and $\phi_a \to (\phi_a)^i{}_j$. Therefore, the gauge theory becomes $U(N)$ gauge theory. The diagonal components of the scalar fields $(\phi_a)^i{}_i$ correspond to the transverse fluctuations of the $m^{\rm th}$ individual branes. Although the direct way to obtain the effective action of such a system is to generalize the BI action (2.146) for nonabelian case, it is not easy to write down this explicitly. Alternatively, under the small field strength approximation, we generalize the Yang-Mills part of the action (2.149) as

$$S^{\rm eff}_{{\rm D}p-{\rm brane}} = -\frac{1}{g_{\rm YM}^2}\int_\Sigma {\rm d}^{1+p}\sigma\ {\rm Tr}\Big\{\frac14 F_{mn}F^{mn} + \frac12 D_m\phi_a D^m\phi^a - \frac14\sum_{a\neq b}[\phi^a,\phi^b]^2\Big\}, \tag{2.153}$$

where we have defined the covariant derivative as $D_m\phi^a = \partial_m\phi^a + i[A_m,\phi^a]$. The structures of the commutator could be again understood from T-duality perspective. We obtained $U(N)$ Yang-Mills (+Higgs) theory in $(1+p)$-dimensional world volume from the $N\times$Dp-branes effective action. The overall $U(1)$ factor of $U(N)$ corresponds to the overall position of the branes and may decouple from the $SU(N)$ gauge dynamics.

[16] Open strings which we discuss have an orientation. A string starting from D-brane i and ending on j is different from the one starting from j and ending on i.

Let us discuss Higgs mechanism in D-brane picture [34]. As usual, by using gauge transformation, we could find the potential minimum in the configuration

$$\phi^a = \mathrm{diag}(v_1^a, \cdots, v_N^a). \tag{2.154}$$

The diagonal component v_m^a represents the location of the m^{th} D-brane in the transverse directions. N D-branes sit on the same position, $U(N)$ symmetry holds. If all of the D-branes are separated, the VEV (2.154) breaks the gauge symmetry $U(N) \to U(1)^N$ so that we could expect W-bosons to get mass. The essential mechanism can be easily understood in $U(2)$ case. For simplicity we consider a D-brane that has VEV only in one transverse direction X^D; $\phi^D = \mathrm{diag}(v_1, v_2)$. The gauge field A_m can be expressed as

$$A_m = \begin{pmatrix} A_m^1 & W_m \\ W_m^* & A_m^2 \end{pmatrix}.$$

Through the covariant derivative in the action (2.153) W-boson obtains a mass term

$$(v_1 - v_2)^2 |W_m|^2 = \frac{1}{(2\pi\alpha')^2}(X_1^D - X_2^D)^2 |W_m|^2 = \frac{1}{(2\pi\alpha')^2}(\text{string length})^2 |W_m|^2, \tag{2.155}$$

while two gauge field $A_m^{(1,2)}$ remain massless. As we observed in (2.110b), the mass can be provided as the mass of open string which is stretched between two D-branes.

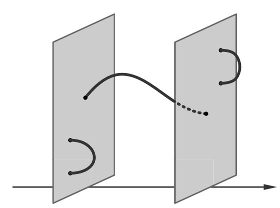

Figure 9: Stringy Higgs mechanism

2.6.2 Dp-branes and RR p-brane solitons in type II supergravity

In order to compare the Dp-brane with RR p-brane, we need to know the charge μ_p in (2.152) carried by a single Dp-brane. In analogy with point charges, we consider the force between D-branes.

We put two parallel flat D-branes with distance $|\vec{Y}|$. The total force can be estimated through the exchanges of closed string states (See Fig.10). We calculate this amplitude by two ways and compare

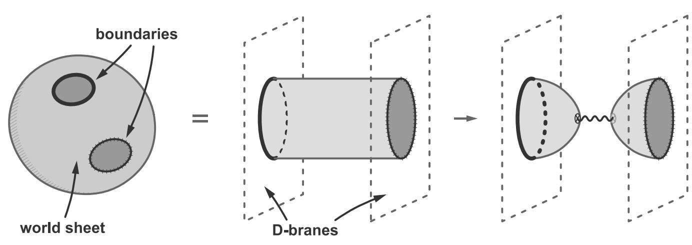

Figure 10: Interaction between Dp-branes

the results. The first one is that from pure perturbative string theory point of view; the diagram can be visualized as a cylinder where closed strings propagate between two D-branes. However, in the world sheet point of view, we can interpret this as an open string one loop amplitude (cf. (2.53)),

$$\begin{aligned} \mathcal{A}_{1-\mathrm{loop}} &= -\frac{1}{2}\log \mathrm{Det}(\text{propagator})^{-1} = \frac{1}{2}\int_0^\infty \frac{\mathrm{d}t}{t}\mathrm{Tr}\, \mathrm{e}^{-tL_0} \sim 2i\pi(1-1)(4\pi^2\alpha')^{3-p} V_{1+p} G_{9-p}(|\vec{Y}|), \\ &= \mathcal{A}_{\mathrm{NSNS}} + \mathcal{A}_{\mathrm{RR}}, \end{aligned} \tag{2.156}$$

where V_{1+p} is the volume of p-brane and $G_d(x)$ is a d-dimensional massless Green's function[17]

$$G_d(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{e}^{ikx}}{k^2} = \frac{\Gamma\big((d-2)/2\big)}{4\pi^{d/2}|x|^{d-2}}.$$

In (2.156) we have extracted the contributions from long range interactive massless modes in closed string. Here, $\mathcal{A}_{\mathrm{NSNS}} = 2i\pi(4\pi^2\alpha')^{3-p}V_{1+p}G_{9-p}(|\vec{Y}|)$, $\mathcal{A}_{\mathrm{RR}} = -\mathcal{A}_{\mathrm{NSNS}}$. The absence of the string coupling g_{s} in (2.156) reflects that the amplitude has been computed on the world sheet with the cylinder topology. So we observe that the force between Dp-branes totally vanishes. This can be interpreted as the cancellation between the attractive forces due to graviton and dilaton and the repulsive forces from the RR gauge fields. In (2.156), $+1$ and -1 arise from gravity sector and RR gauge sector, respectively. Resulting stable configuration in fact indicates that D-branes are BPS states.

Next, we calculate the same amplitude from the effective theory point of view. The ingredients here are the NSNS graviton sector and RR sector everywhere, the gauge fields on Dp-brane, and the scalar fields $\phi_a(\sigma)$. Since we are interested in the flat Dp-brane configuration, the embedding functions $X^M(\sigma)$ have a trivial meaning that the scalar fields $\phi_a(\sigma)$ are constants which just specify the location of the Dp-branes. We perturb the NSNS-RR background fields through (2.148) and $C_{p+1}(x) = 0 + \kappa_0\widehat{C}_{p+1}(x)$, and observe responses of Dp-branes. The relevant source/linear coupling terms can be found in the DBI action (2.149) and the WZ term (2.152),

$$S_{\mathrm{DBI}}^{\mathrm{source}} = -T_p\kappa\int_{\Sigma_{p+1}} \mathrm{d}^{1+p}\sigma\Big(\frac{p-3}{4}\widehat{\Phi} + \frac{1}{2}\widehat{h}_m{}^m\Big), \quad S_{\mathrm{WZ}}^{\mathrm{source}} = -\mu_p\kappa_0\int_{\Sigma_{p+1}}\widehat{C}_{p+1} = -\mu_p\kappa_0\int_{\Sigma_{p+1}}\mathrm{d}^{1+p}\sigma\ \widehat{C}_{01\cdots p}. \tag{2.158}$$

As we have already mentioned, the antisymmetric Kalb-Ramond field $\widehat{b}_{mn}(\sigma)$ and also the gauge field strength $F_{mn}(\sigma)$ do not couple linearly. In the WZ term, a component survived is only $\widehat{C}_{01\cdots p}(x)$ in the flat embedding. The bulk propagators which couple to the D-brane can be read off from the action (2.131) under the perturbation (2.148)[18],

$$\langle\widehat{\Phi}(-k)\widehat{\Phi}(k)\rangle = -\frac{2i}{k^2}, \qquad \langle\widehat{C}_{01\cdots p}(-k)\widehat{C}_{01\cdots p}(k)\rangle = \frac{2i}{k^2}.$$

$$\langle\widehat{h}_{MN}(-k)\widehat{h}_{KL}(k)\rangle = -\frac{2i}{k^2}\Big(\eta_{MK}\eta_{NL} + \eta_{ML}\eta_{NK} - \frac{1}{4}\eta_{MN}\eta_{KL}\Big), \tag{2.159}$$

We are ready to compute the tree amplitudes for each sector. The result is as follows for RR gauge field sector and NSNS gravity sectors, respectively,

$$\mathcal{A}_{\mathrm{RR}} = (i\mu_p\kappa_0)^2 V_{1+p}\int\frac{\mathrm{d}^{9-p}k}{(2\pi)^{9-k}}\frac{2i}{k^2}\ \mathrm{e}^{i\vec{k}\vec{Y}} = -2i\mu_p^2\kappa_0^2V_{1+p}G_{9-p}(|\vec{Y}|). \tag{2.160}$$

$$\begin{aligned}\mathcal{A}_{\mathrm{NSNS}} &= (iT_p\kappa)^2V_{1+p}\int\frac{\mathrm{d}^{9-p}k}{(2\pi)^{9-p}}(-\frac{2i}{k^2})\bigg\{\Big(\frac{p-3}{4}\Big)^2 + \Big(\frac{1}{2}\Big)^2\Big(2\eta_m{}^n\eta_n{}^m - \frac{1}{4}\eta_m{}^m\eta_n{}^n\Big)\bigg\}\ \mathrm{e}^{i\vec{k}\vec{Y}}\\ &= 2i\kappa_0^2g_{\mathrm{s}}^2T_p^2V_{1+p}G_{9-p}(|\vec{Y}|).\end{aligned} \tag{2.161}$$

Comparing the previous result (2.156), we can obtain relations $\mu_p^2\kappa_0^2 = \pi(4\pi^2\alpha')^{3-p}$ and $\kappa_0^2g_{\mathrm{s}}^2T_p^2 = \pi(4\pi^2\alpha')^{3-p}$, and the BPS condition

$$\mu_p = g_{\mathrm{s}}T_p = \frac{1}{(2\pi)^p\alpha'^{\frac{p+1}{2}}}, \tag{2.162}$$

[17]For later references, we give the formula for the Fourier transformation,

$$\int\frac{\mathrm{d}^dk}{(2\pi)^d}\ \mathrm{e}^{ikx}k^n = \frac{2^n}{\pi^{d/2}}\frac{\Gamma\big(\frac{d+n}{2}\big)}{\Gamma\big(-\frac{n}{2}\big)}\frac{1}{|x|^{d+n}}. \tag{2.157}$$

[18]For the graviton sector, we refer to the propagator (2.7).

where for the last equality we have used (2.147). The condition (2.162) could be a counter part of the equality in (2.141). Moreover we can fix the 10D Newton constant in (2.132) in terms of the parameter in string theory,

$$G_{\rm N}^{(10)} = \kappa_0^2 g_{\rm s}^2/(8\pi) = 8\pi^6 g_{\rm s}^2 \alpha'^4 \equiv \kappa^2/(8\pi), \qquad \text{i.e.} \quad \kappa = 2\pi^{3/2} g_{\rm s}(2\pi\alpha')^2. \tag{2.163}$$

We can also show that the charge is quantized. Comparing the normalization factors of the kinetic terms between (2.58) and (2.134), we can conclude the Dirac quantization condition (2.64) is satisfied with $n = 1$,

$$\widetilde{\mu}_p \widetilde{\mu}_{6-p} = 2\pi, \tag{2.164}$$

where we have rescaled the charge (2.162) as $\widetilde{\mu}_p \equiv \sqrt{2}\kappa_0 \mu_p$.

We have observed that Dp-brane and the extremal RR p-brane share the same properties. Both of them correspond to BPS states and possess RR charges. This suggests that the Dp-brane and the extremal RR p-brane could be identified. Under the identification of charges (2.137) and (2.162),

$$Q_p = N \times \mu_p, \qquad \text{i.e.} \qquad \int_{S^{8-p}} *F_{p+2} = \frac{N}{(2\pi\sqrt{\alpha'})^{p-7}}, \tag{2.165}$$

where N is the number of coincident Dp-branes[19], the parameter L corresponding to the Schwarzschild radius can be fixed as

$$L^{7-p} = \frac{(2\pi)^{7-p} \alpha'^{\frac{7-p}{2}} N g_{\rm s}}{(7-p) V_{S^{8-p}}}. \tag{2.166}$$

However, the Dp-brane and RR p-brane pictures are reliable in different regimes. If one introduces $N\times$ Dp-branes in string perturbation theory, the effective string coupling is raised to be $N \times g_{\rm s}$ according to the open string loop counting (2.78). Therefore, in the string perturbation theory, the condition

$$\lambda \equiv N g_{\rm s} \ll 1 \tag{2.167}$$

should be required. In the bulk spacetime point of view, the condition (2.167) is equivalent to

$$r_{\rm S} \ll \sqrt{\alpha'}, \tag{2.168}$$

where $r_{\rm S} = \left(\frac{7-p}{8-p}\right)^{1/(7-p)} L$ is the Schwarzschild radius. The gravitational scale is much smaller than the string scale, so that the physics with the length scale $r_{\rm S}$ is not affected by strings. The gravitational backreaction of D-brane are arbitrary small, and D-brane does not modify the geometry in the perturbative regime. This can be also understood in the Einstein equation (2.3) by computing the energy momentum tensor ($\sim$ tension) from Dp-brane:

$$R_{tt} - \frac{1}{2} G_{tt} R = 8\pi G_{\rm N}^{(10)} T_{tt} \propto g_{\rm s}^2 \times \left(N \times \frac{1}{g_{\rm s}}\right) \propto \lambda. \tag{2.169}$$

Therefore, the D-brane and its open string modes are defined in the flat spacetime.

On the other hand, RR p-branes are nontrivial solutions of type II supergravity actions which are the low energy effective action of the closed string. As we discussed, the low energy effective action itself can be obtained under the weak curvature approximation measured by the string unit (2.84), i.e.

$$r_{\rm S} \gg \sqrt{\alpha'}, \tag{2.170}$$

[19] As BPS configuration indicates, in general, there also exist multi-center solutions in the supergravity which correspond to parallel branes separated by some distance.

which means the Schwarzschild radius is now macroscopic. The condition (2.170) is equivalent to

$$\lambda \gg 1. \tag{2.171}$$

According to (2.169), in this case, D-brane would gravitationally collapse to the RR p-brane soliton. It should be noted that the number N should be large in order to keep the condition (2.171) under the string perturbation $g_s \ll 1$.

These observations suggest the "duality" between two pictures i.e. Dp-brane and RR p-brane soliton pictures, in the different limits of the parameter λ.

3 AdS/CFT correspondence

The AdS/CFT correspondence is a (gravity theory in AdS spacetime) /(conformal field theory) duality in string theory [1] (for review see [35]). This correspondence is based on two ideas. One is holography which relates theories living in spaces with different dimensionality. Inspired by the "area low" of black hole entropies, the holography is introduced as the statement: quantum gravity in some region may be described in terms of a non gravitational theory living on its boundary [36]. Another important idea is that the nonabelian gauge theory might have a dual description in terms of strings [37], i.e. the 't Hooft large N expansion. These two can be realized with precise meaning through the AdS/CFT correspondence. Despite the name, the correspondence can be naturally extended to non AdS spacetime and non-conformal gauge field theories.

3.1 Maldacena conjecture

In the last section, the Dp-branes and the RR p-brane solutions in supergravity theories describe the same BPS states. Pictorial identification between D3-brane and extremal RR-3-brane is given by Fig.11. By looking into the detail of the two descriptions and taking certain limits, we can obtain a new insight:

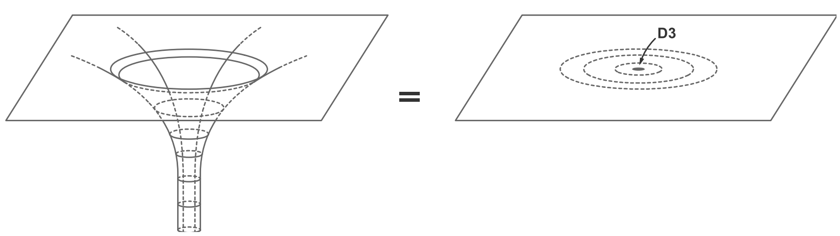

Figure 11: RR p-brane and Dp-brane

the AdS/CFT correspondence. As the original and the concrete example, we start with N_c coincident D3-branes in type IIB superstring theory in 10D.

Let us first consider N_c D3-branes in string perturbation theory where there exist open strings ending on these D3-branes and also closed strings in the bulk. At the low energy

$$\omega \ll 1/\sqrt{\alpha'} \propto 1/\kappa^{1/4}, \tag{3.1}$$

we may obtain the low energy effective action

$$S = S_{\text{bulk}} + S_{\text{D3-branes}} + S_{\text{interactions}}. \tag{3.2}$$

The bulk action is that of 10D supergravity which describes the massless closed string excitation modes (2.131), and some derivative corrections. The D-brane actions are given by (2.146), whose α' expansion

is (2.149), and (2.152). As we mentioned, the D-brane effective action contains the Yang-Mills gauge theory with coupling (2.151), i.e. for D3-branes,

$$g_{\mathrm{YM}}^2 = 2\pi g_{\mathrm{s}}. \tag{3.3}$$

In the case of the N_{c} D3-branes, its 4D world volume gauge theory is $\mathcal{N}=4$ $SU(N_{\mathrm{c}})$ supersymmetric Yang-Mills theory whose bosonic part is the same as (2.153)[20]. The interaction part $S_{\mathrm{interactions}}$ may be generated by integrating out the higher energy modes of string excitations. These are typically proportional to the positive power of the gravitation constant κ.

Let us now focus on the 4D super Yang-Mills theory. First of all, as we have observed in (2.149), in general, the super Yang-Mills couples to the bulk modes including graviton. If we are interested in the super Yang-Mills gauge theory itself, we need to decouple these bulk gravity modes. This could be achieved by taking the limit; the gravitational constant $\kappa \to 0$ while keeping the Yang-Mills coupling constant g_{YM} finite. Looking at the couplings (2.132) and (3.3), this is equivalent to the limit $\alpha' \to 0$ with fixed g_{s}. In this limit with fixed energies all of the interactions vanish except for BPS condition. For the gravity part, the action simply follows (2.6) in 10D. Therefore, the total system falls into the two independent ones, i.e. 4D pure $\mathcal{N}=4$ $SU(N_{\mathrm{c}})$ super Yang-Mills on the D3-brane and free type IIB supergravity in the 10D bulk. This is so-called decoupling limit. Schematically,

$$\begin{aligned}&(\text{D3-branes in flat 10D Minkowski spacetime})\\ &\qquad \text{—— (low energy limit)} \longrightarrow \begin{cases}(\mathcal{N}=4 \text{ super Yang-Mills in 4D world volume})\\ (\text{decoupled 10D supergravity on flat spacetime})\end{cases}\end{aligned} \tag{3.4}$$

It is worth to mention the energy scale in the gauge theory side described by open strings [1]. To take the limit $\alpha' \to 0$, in general, we need to fix the energy scale. Below the string energy scale, the natural scale may be the mass of W-boson (2.155),

$$\omega_W \equiv \frac{r}{\alpha'}, \tag{3.5}$$

where r is the distance between D-branes[21]. Since the low energy dynamics of the gauge theory are described by the open strings attached to the D-branes, it would be natural to take the Higgs mechanism as a typical source of the energy scale. Keeping the energy (3.5) finite and taking the limit $\alpha' \to 0$ requires the scaling behavior $r \propto \alpha' \to 0$. This indicates that an observer should be located far from the D3-branes. The low energy limit (3.1) directly suggests

$$r \ll \sqrt{\alpha'}, \tag{3.6}$$

which will be realized as the near horizon limit (2.144) in the dual geometry.

We shall consider the same N_{c} D3-branes system from the point of view of type IIB superstring on the solitonic background (2.143) with

$$L^4 = 4\pi\alpha'^2 N_{\mathrm{c}} g_{\mathrm{s}}, \tag{3.7}$$

which comes from the identification (2.166). Since the geometry possesses the asymptotic flat region, it is natural to see physics from that region as we eventually did in the previous perturbative string picture.

[20]D-brane preserves 16 supercharge as the BPS states. In 4D these supercharges can be decomposed into four Weyl spinors. Thus the low energy effective action of D3-brane should have $\mathcal{N}=4$ supersymmetries.

[21]This can be realized as the following situation; we put the stack of $(N_{\mathrm{c}}-1)$ D3-branes at the origin and introduce the probe single D3-brane at the position r. By definition, the probe does not do anything to the others. The decoupling limit implies that we could only observe the stack of D3-branes at the origin.

In the background (2.143), due to the gravitational red shift (2.25), the observer in the asymptotic region will measure an energy ω_∞ which is smaller than the original energy ω_r emitted from the position r,

$$\omega_\infty = \sqrt{-G_{tt}(r)}\ \omega_r = \Big(1+\frac{L^4}{r^4}\Big)^{-\frac{1}{4}}\omega_r. \tag{3.8}$$

For $r > L$, the low energy excitations are those of free type IIB supergravity. In the region $r < L$, the red shift factor becomes more important, especially in the near horizon $r \ll L$, the energy is measured as

$$\omega_\infty \sim \frac{r}{L}\omega_r\,. \tag{3.9}$$

This implies that the "finite" energy modes near horizon ($r \ll L$) may be observed as the low energy modes in the asymptotic region. Indeed string excitation modes $\omega \sim 1/\sqrt{\alpha'}$ could be observed as

$$\omega_\infty \sim \frac{r}{L}\omega_r \propto \frac{r}{\sqrt{\alpha'}}\frac{1}{\sqrt{\alpha'}} = \frac{r}{\alpha'}\,, \tag{3.10}$$

where we have used the relation (3.7). This could be the energy scale which should be fixed during the limit $\alpha' \to 0$ which we have discussed as the decoupling limit in the gauge theory. Indeed the fixed energy scale (3.10) is the same with (3.5). The low energy decoupling limit (3.6) also supports the near horizon region i.e. $r \ll \sqrt{\alpha'} \sim L$.

As a result, low energy excitations at infinity can be categorized into two types, i.e. the low energy excitations far from the branes and finite energy excitations near horizon.

It is important to see that these excitations could be decoupled each other. As a typical experiment at the asymptotic region, we consider the graviton scattering with the excitations on the D3-branes in the low energy. The characteristic behavior of the absorption cross section of the D3-branes can be found with the incident energy ω as [38]

$$\sigma_{\mathrm{D3}} \propto \omega^3 L^8 \propto \omega^3 \alpha'^4 N_{\mathrm{c}}^2 g_{\mathrm{s}}^2, \tag{3.11}$$

where the degrees of freedom for the gauge field N_{c}^2 show up. This indicates that, in low energies $\omega \ll 1/\sqrt{\alpha'}$ with fixed g_{s} and N_{c}, D3-branes do not interact with supergravity excitations in the asymptotic region. In other words, since the low energy long wave length $1/\omega \gg \sqrt{\alpha'}$ cannot enter the small throat $L \sim \sqrt{\alpha}$, they could not interact each other. Therefore, we conclude that in the solitonic background picture, the excitation modes measured in the low energy could be well approximated by two isolated parts, i.e. those for the 10D bulk free supergravity and for the type IIB superstring in the near horizon region.

As we did in the perturbative string picture, we shall take the low energy limit $\alpha' \to 0$ with fixed energies and fixed dimensionless quantities. Keeping the energy (3.10) fixed, in this energy scale, the metric (2.143) becomes the near horizon metric

$$\frac{\mathrm{d}s^2}{\alpha'} = \frac{U^2}{\sqrt{4\pi N_{\mathrm{c}} g_{\mathrm{s}}}}\Big(-\mathrm{d}t^2 + \sum_{i=1}^{3}\mathrm{d}x^{i2}\Big) + \frac{\sqrt{4\pi N_{\mathrm{c}} g_{\mathrm{s}}}}{U^2}\mathrm{d}U^2 + \sqrt{4\pi N_{\mathrm{c}} g_{\mathrm{s}}}\mathrm{d}\Omega_5^2, \tag{3.12}$$

where we have defined the coordinate $U = r/\alpha'$.

Another natural energy scale has been considered by the probe of the field in supergravity to the stack of D3-branes [39]. The typical energy scale of the supergravity field in (3.12) becomes $u = r/L^2 (\propto r/\alpha')$. In this energy scale, the near horizon metric becomes

$$\frac{\mathrm{d}s^2}{L^2} = u^2\Big(-\mathrm{d}t^2 + \sum_{i=1}^{3}\mathrm{d}x^{i2}\Big) + \frac{1}{u^2}\mathrm{d}u^2 + \mathrm{d}\Omega_5^2. \tag{3.13}$$

These metrics (3.12) and (3.13) describe $\mathrm{AdS}_5 \times S^5$ geometry. Therefore schematically,

$$\begin{aligned}&\text{(D3-branes as the solitonic background)}\\ &\quad\text{—— (low energy limit)} \longrightarrow \begin{cases}\text{(excitations in } \mathrm{AdS}_5 \times S^5 \text{ region)}\\ \text{(decoupled 10D supergravity on flat spacetime)}\end{cases}\end{aligned} \tag{3.14}$$

Comparing these two pictures in the low energy (3.4) and (3.14), Maldacena conjectured that

$$(\mathcal{N} = 4 \text{ super Yang-Mills in 4D}) \sim (\text{type IIB string theory on } \mathrm{AdS}_5 \times S^5) \tag{3.15}$$

We discuss this conjecture more precisely in the next subsection.

3.2 Correspondence

3.2.1 Symmetries

In the field theory side, i.e. 4D $\mathcal{N} = 4$ super Yang-Mills theory, field contents in the adjoint representation are massless gauge fields, 6 real massless scalars, and four massless Weyl fermions as gaugino. Under $SU(4)$ R-symmetry transformation, which is a global symmetry for 4D $\mathcal{N} = 4$ supersymmetry with 16 supercharges, the scalars and fermions transform as **6** and **4**, respectively. It can be shown that the beta function vanishes at least 3 loops and believed to be vanished in all orders of the perturbation. This feature indicates that the 4D $\mathcal{N} = 4$ super Yang-Mills theory is a conformal field theory.

Let us discuss the relation between the conformal symmetry in the gauge theory and the isometry (symmetries of the metric) of AdS spacetime. The conformal group in D dimensional spacetime is the group of reparameterizations which preserve the spacetime metric up to a local scale factor,

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = \mathrm{e}^{\Lambda(x)} g_{\mu\nu}(x). \tag{3.16}$$

The conformal transformations can be generated by the infinitesimal coordinate transformations $x^\mu \to x^{\mu'} = x^\mu + \varepsilon^\mu(x)$ which satisfy

$$\delta g_{\mu\nu}(x) = g'_{\mu\nu}(x) - g_{\mu\nu}(x) = \nabla_\mu \varepsilon_\nu + \nabla_\nu \varepsilon_\mu = \Lambda(x) g_{\mu\nu}(x). \tag{3.17}$$

Taking the trace of the both sides, one can eliminate the factor $\Lambda(x)$ and obtain the conformal Killing equation

$$\nabla_\mu \varepsilon_\nu + \nabla_\nu \varepsilon_\mu = \frac{2}{D} g_{\mu\nu}(g^{\rho\sigma} \nabla_\rho \varepsilon_\sigma). \tag{3.18}$$

In flat spacetime ($D \geq 3$), one can solve the equation (3.18) and obtain the generator $\varepsilon^\mu(x)$ of the form:

$$\varepsilon^\mu(x) = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + 2(c_\lambda x^\lambda) x^\mu - c^\mu(x_\lambda x^\lambda). \tag{3.19}$$

Apart from usual D-dimensional Poincaré symmetries $ISO(1, D-1)$, i.e. translation and rotation which are represented by the constant parameters a^μ and $\omega_{\mu\nu} = -\omega_{\nu\mu}$, respectively, we obtain the dilatation and the conformal boost whose infinitesimal transformation parameters are given by λ and c^μ. Integrating the infinitesimal transformations, the finite transformations could be obtained. In addition to usual Poincaré transformations, we could arrive at those for the dilatation and the conformal boost,

$$x'^\mu = \mathrm{e}^{-\lambda} x^\mu, \qquad x'^\mu = \frac{x^\mu - c^\mu x^2}{1 - 2(cx) + c^2 x^2}. \tag{3.20}$$

All of these transformations form the conformal group which is isomorphic to the $SO(2, D)$.

In the gravity side, we solve the Killing equation

$$\delta g_{MN} = \nabla_M \varepsilon_N + \nabla_N \varepsilon_M = 0, \tag{3.21}$$

to find symmetries of the metric. We are interested in the AdS part of the metric (3.13),

$$\mathrm{d}s^2 = L^2\Big(u^2\big(-\mathrm{d}t^2 + \sum_{i=1}^{3}\mathrm{d}x^{i2}\big) + \frac{1}{u^2}\mathrm{d}u^2\Big) = L^2\Big(u^2\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \frac{1}{u^2}\mathrm{d}u^2\Big). \tag{3.22}$$

Then, in this coordinate system, we find the following generators ($\delta x^M = \varepsilon^M$ with $x^M = (x^\mu, u)$) which are isomorphic to $SO(2,4)$; the translation $\delta x^\mu = a^\mu$, $\delta u = 0$, the rotation $\delta x^\mu = \omega^\mu{}_\nu x^\nu$, $\delta u = 0$, the dilatation

$$\delta x^\mu = -\lambda x^\mu, \tag{3.23a}$$
$$\delta u = \lambda u, \tag{3.23b}$$

and the conformal boost

$$\delta x^\mu = 2(c_\nu x^\nu)x^\mu - c^\mu(x_\nu x^\nu) - \frac{c^\mu}{u^2}, \tag{3.24a}$$
$$\delta u = -2(c_\nu x^\nu)u. \tag{3.24b}$$

It is interesting to see that the transformation (3.24a) becomes the conformal boost in 4D (conformally) flat spacetime if the limit $u \to \infty$ is taken. Thus, the conformal symmetry in the 4D gauge theory appears in the region $u \to \infty$ which is the boundary of the AdS spacetime in the dual gravity theory. The isometry group of AdS_5 acts as the conformal group on its 4D boundary. This is a realization of the holography.

The dilatation (3.23a) and (3.23b) or their finite forms of the transformations $(x^\mu, u) \to (\mathrm{e}^{-\lambda}x^\mu, \mathrm{e}^{\lambda}u)$ with a constant λ indicate that the radial coordinate u can be understood as the energy scale of the boundary ($u \to \infty$) 4D theory [39]. Thus, the holographic dual may correspond to the conformal field theory in the UV.

In the gauge theory side, there exists $SU(4)$ R-symmetry. In the gravity side, there also exists $SO(6) \simeq SU(4)$ symmetry which is isometry group of S^5 part of the geometry. Thus, we conclude that the global symmetries of $\mathcal{N} = 4$ super Yang-Mills theory match with the isometries of the $\mathrm{AdS}_5 \times S^5$ background. This matching of the symmetry is one of the most powerful checks of the correspondence. Indeed, the symmetry matching plays an important role later.

3.2.2 Coupling constants

The claim (3.15) without any restrictions on the parameter space is known as the strongest conjecture. Unfortunately, it is not easy to analyze the full type IIB string theory in the near horizon background geometry through conventional approaches. We need to consider suitable approximations to make the problems under control in the current knowledge. Nevertheless, we could deduce many nontrivial properties from the correspondence.

We here consider the relations between parameters of the two theories. In Yang-Mills theory, there are two parameters i.e. the gauge coupling g_{YM} and N_{c} for the gauge group $SU(N_{\mathrm{c}})$. In the string theory side, we have the string tension α' and the string coupling g_{s}. These are related through (3.3) and (3.7),

$$g_{\mathrm{YM}}^2 = 2\pi g_{\mathrm{s}} \quad \text{and} \quad N_{\mathrm{c}} = \frac{L^4}{4\pi g_{\mathrm{s}}\alpha'^2}. \tag{3.25}$$

The parameter introduced in (2.167) is now interpreted as 't Hooft coupling $\lambda = N_\mathrm{c} g_\mathrm{s} \propto N_\mathrm{c} g_\mathrm{YM}^2$ which is a relevant loop expansion parameter rather than g_YM in the large N gauge theory [37]. In the gravity picture, (3.14) is applicable for the large 't Hooft coupling (2.171) and also large N_c for the string perturbation theory. By using the AdS/CFT argument, the corresponding gauge theory is a strongly coupled large N_c Yang-Mills theory. This opens a new window to analyze the strongly correlating systems by using the dual gravity theory via AdS/CFT correspondence. We could describe the strongly coupled gauge theories in terms of the weakly coupled classical gravity theories.

In general, it is hard to "prove" such a strong/weak duality mainly due to the lack of nonperturbative techniques. So far a huge number of "tests" for the correspondence have been worked out and no contradiction has been found. However, for tightly protected parts by the symmetry, we could verify the duality. BPS states are good for this purpose, because, for instance, expectation values of these do not depend on the coupling constant λ. We could directly compare two different theories.

3.2.3 Partition function

Since two pictures, i.e. gauge/gravity, describe the same physical system, the responses to perturbations should be same.

Quantitative correspondence can be given by the path integral formulation of the two theories [2,3]. Partition functions in two different pictures should be equated in the AdS/CFT correspondence:

$$\begin{aligned} Z_\mathrm{CFT}(\text{4D super Yang-Mills}) &\equiv Z_\mathrm{string}(\text{IIB on AdS}_5 \times S^5) \\ &\sim Z(\text{IIB supergravity on AdS}_5 \times S^5) \\ &\sim \mathrm{e}^{-S_\mathrm{supergravity}}\Big|_\text{on-shell} \qquad (N_\mathrm{c}, \lambda \gg 1), \end{aligned} \tag{3.26}$$

where in the string side, we may evaluate the partition function as the saddle point approximation of the supergravity action with the large N_c and the large 't Hooft coupling. For definiteness, we have equated the relation in Euclidean signature.

Let us consider perturbations or excitations with a precise example. We perturb the fields in the gravity picture which originate from the type IIB superstring. The response to the gauge field theory, for instance, could be observed in the perturbed action (2.149). The perturbations produce the following interaction terms which are added to the kinetic terms of the gauge theory,

$$S_\mathrm{int} = -T_3(2\pi\alpha')^2\kappa \int \mathrm{d}^4x \Big(-\frac{1}{4}\widehat{\Phi} F_{mn}^2 + \frac{1}{2}\widehat{h}_{mn} T^{mn} \Big). \tag{3.27}$$

After taking the decoupling limit, the fields in the gravity sectors are regarded as the external fields which are not dynamical in the gauge field side. These external fields act as the sources which couple to operators in the gauge field theory. In the interactions (3.27), as the sources, the dilaton $\widehat{\Phi}(x^m)$ and the metric perturbations $\widehat{h}_{mn}(x^m)$ couple to the glueball $F_{mn}^2(x^m)$ and the energy momentum tensor $T^{mn}(x^m)$ which would be operators in the gauge field theory. Therefore under the perturbations, the partition function of the gauge field theory turns to be the generating function of the correlators of some operators.

The sources $\widehat{\Phi}(x^m)$ and $\widehat{h}_{mn}(x^m)$ originate from the perturbation of the fields in the bulk. Thus, from the bulk gravity theory point of view, the source defined on the D3-brane could be a boundary value of the field in the bulk spacetime,

$$\widehat{\Phi}(x^m) = \widehat{\Phi}(x^M)\Big|_\text{boundary} \qquad \text{and} \qquad \widehat{h}_{mn}(x^m) = \widehat{h}_{mn}(x^M)\Big|_\text{boundary}. \tag{3.28}$$

In the bulk gravity theory we need to solve classical equations of motion for $\widehat{\Phi}(x^M)$ and $\widehat{h}_{mn}(x^M)$ to evaluate the on-shell action. The condition (3.28) can be understood as the boundary conditions for these differential equations.

We could apply the idea to a more general case. There may exist maps between the fields $\varphi(x^M)$ in the bulk gravity side and the local gauge invariant operators $\mathcal{O}(x^m)$ in the 4D gauge theory side. With the argument above, the practical form of the conjecture (3.26) becomes

$$\mathrm{e}^{W[\varphi_0]} \equiv \left\langle \, \mathrm{e}^{\int \mathrm{d}^4 x \varphi_0(x^m)\mathcal{O}(x^m)} \, \right\rangle_{\mathrm{CFT}} = \mathrm{e}^{-S_{\text{supergravity}}[\varphi_0(x^m)]}\Big|_{\text{on-shell}} \quad \text{with} \quad \varphi_0(x^m) = \varphi(x^M)\big|_{\text{boundary}}, \quad (3.29)$$

which is known as Gubser-Klebanov-Polyakov-Witten (GKP-W) relation. $W_{\mathrm{CFT}}[\varphi_0]$ is the generating functional for connected diagrams. Suppose we have obtained a solution of the equation of motion in the bulk spacetime with required boundary conditions. We could compute the on-shell action by using the solution to express the on-shell action in terms of the boundary value which is the source in the field theory point of view. By differentiating the relation (3.29) with respect to the source $\varphi_0(x^\mu)$, we could obtain the correlation function for corresponding operator in the gauge theory side,

$$\big\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\big\rangle = (-)^{n+1}\frac{\delta}{\delta\varphi_0(x_1)}\cdots\frac{\delta}{\delta\varphi_0(x_n)}S_{\text{supergravity}}[\varphi_0]\big|_{\text{on-shell}}\Big|_{\varphi_0=0}. \quad (3.30)$$

In principle, by using the complete set of the Kaluza-Klein mass spectrum of 10D type IIB supergravity on S^5 [32,40], we could analyze the dynamics of the perturbations around $\mathrm{AdS}_5\times S^5$ background. However, one of the obstructions comes from the nature of the $\mathrm{AdS}_5\times S^5$ background. Since the radius of S^5, which is the same with that of AdS_5, cannot be arbitrary small, we have to tackle with complicated Kaluza-Klein modes. The next procedure we should pursue is to find out consistent truncations of the uncontrollable modes. However, except for a few sectors [41], this direction is also complicated.

As an alternative, one could start from 5D theory. There is a complete 5D supergravity theory known as the $SO(6)$ gauged $\mathcal{N}=8$ supergravity which contains the same supergravity multiplet as the type IIB supergravity [42]. It is widely accepted that this supergravity theory may be a consistent truncation of the type IIB theory on $\mathrm{AdS}_5\times S^5$. We here assume these effective 5D theories.

Before calculating correlation functions a la AdS/CFT correspondence (3.30) through the gravitational theory, let us briefly summarize the conformal field theory side in 4D. In usual quantum field theories, we classify local fields and operators $\mathcal{O}(x^\mu)$ by their representations of the Poincaré group, while in conformal field theories, we use the conformal group $SO(2,4)$. The conformal group has the little group $SO(1,1)\times SO(1,3)$, more conveniently compact group $SO(2)\times SO(4)$, so that in addition to the spin representation, we have the scaling dimension Δ which is the charge of $SO(2)$ subgroup. We define the scaling dimension of a operator $\mathcal{O}_\Delta(x^\mu)$ as

$$\mathcal{O}_\Delta(x^\mu) \to \mathcal{O}'_\Delta(x'^\mu) = \mathrm{e}^{\lambda\Delta}\mathcal{O}_\Delta(x), \quad (3.31)$$

under the scaling $x^\mu \to x'^\mu = \mathrm{e}^{-\lambda}x^\mu$. Conformal invariance determines the forms of the correlators of the primary fields in terms of their scaling dimensions. For example, for scalar primary fields, correlators are given by

$$\begin{aligned}
\big\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\big\rangle &= \delta_{1,2}\prod_{i<j}^{2}|x_{ij}|^{-\Delta},\\
\big\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\big\rangle &= c_{123}\prod_{i<j}^{3}|x_{ij}|^{\Delta-2\Delta_i-2\Delta_j},\\
\big\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4)\big\rangle &= c_{1234}(u,v)\prod_{i<j}^{4}|x_{ij}|^{\frac{1}{3}\Delta-\Delta_i-\Delta_j},
\end{aligned} \quad (3.32)$$

where c_{123} and c_{1234} are undetermined coefficients and $x_{ij}=x_i-x_j$ and $\Delta=\Sigma_i\Delta_i$. For scalar operators, the rotational invariance fixes the coordinate dependence in terms of the distance $|x_{ij}|$. Since

the conformal boost in (3.20) gives the following transformation

$$|x'_{ij}|^2 = \frac{|x_{ij}|^2}{\big(1-2(cx_i)+c^2x_i^2\big)\big(1-2(cx_j)+c^2x_j^2\big)},$$

there exist $n(n-3)/2$ conformally invariant ratios $\frac{|x_{ij}||x_{kl}|}{|x_{ik}|x_{jl}|}$ for $n\,(\geq 4)$ point functions. The coefficient c_{1234} in (3.32) depends on these ratios for instance $u=\frac{|x_{12}||x_{34}|}{|x_{13}||x_{24}|}, v=\frac{|x_{14}||x_{23}|}{|x_{13}||x_{24}|}$. Now, let us consider the generating functional of the CFT side in (3.29). The conformal invariance of the action determines the scaling of the source field $\varphi_0(x^\mu)$,

$$\int \mathrm{d}^4x' \varphi'_0(x'^\mu)\mathcal{O}'_\Delta(x'^\mu) = \mathrm{e}^{(-4+\Delta)\lambda}\int \mathrm{d}^4x \varphi'_0(x'^\mu)\mathcal{O}_\Delta(x^\mu), \quad \text{i.e.} \quad \varphi'_0(x'^\mu) = \mathrm{e}^{(4-\Delta)\lambda}\varphi_0(x^\mu). \tag{3.33}$$

This will be used to relate the scaling dimension Δ on boundary CFT with the parameter in the bulk gravity theories.

3.2.4 Correlation functions

It is instructive to discuss the simplest case to understand the heart of the AdS/CFT correspondence. Let us consider a massive scalar field with mass m in 5D AdS spacetime. We here work with the Euclidean Poincaré coordinate of AdS_5 spacetime

$$\mathrm{d}s^2 = \frac{L^2}{z^2}\Big(\delta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}z^2\Big), \tag{3.34}$$

which is given through $u \to L/z$ and $x^\mu \to x^\mu/L$ in the AdS part in (3.13) with Euclidean signature. The boundary and the Killing horizon are now located at $z=0$ and ∞, respectively.

In the AdS_5 background (3.34), the action of the massive scalar $\phi(x)$ with mass m is

$$S = \frac{1}{2}\int \mathrm{d}^5x\sqrt{g}\big(g^{mn}\partial_m\phi\partial_n\phi + m^2\phi^2\big) = \frac{L^3}{2}\int_\epsilon^\infty \mathrm{d}z\int \mathrm{d}^4x\Big(\frac{1}{z^3}\partial_z\phi\partial_z\phi + \frac{1}{z^3}\delta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{L^2m^2}{z^5}\phi^2\Big), \tag{3.35}$$

where we have made the regularization $z=\epsilon\ (\to 0)$. The equation of motion is given by

$$0 = \frac{1}{\sqrt{g}}\partial_m(\sqrt{g}g^{mn}\partial_n\phi) - m^2\phi, \quad \text{i.e.} \quad 0 = \partial_z\Big(\frac{1}{z^3}\partial_z\phi\Big) + \frac{1}{z^3}\delta^{\mu\nu}\partial_\mu\partial_\nu\phi - \frac{L^2m^2}{z^5}\phi, \tag{3.36}$$

where we have imposed the Dirichlet boundary condition at the boundary i.e. $\delta\phi(x)|_{\text{boundary}} = 0$, which is natural choice for AdS/CFT correspondence. Since the 4D Euclidean part is flat, we use the plane wave expansion,

$$\phi(x^\mu, z) = \int \frac{\mathrm{d}^4x}{(2\pi)^4}\,\mathrm{e}^{ik_\mu x^\mu} f_k(z). \tag{3.37}$$

Then, the equation of motion (3.36) becomes the ordinary second order differential equation for the radial coordinate z,

$$0 = f''_k(z) - \frac{3}{z}f'_k(z) - k^2 f_k(z) - \frac{L^2m^2}{z^2}f_k(z), \tag{3.38}$$

where the prime implies the derivative with respect to z. One can solve the equation of motion exactly in this case,

$$f_k(z) = z^2\Big(A(k)K_\nu(kz) + B(k)I_\nu(kz)\Big), \quad \text{with} \quad \nu = \sqrt{4+L^2m^2}. \tag{3.39}$$

Here we consider the case that ν is not an integer. The integration constants $A(k)$ and $B(k)$ should be fixed through suitable boundary conditions. One of these comes from the interior i.e. $z \to \infty$ (IR). Since the Bessel functions $K_\nu(zk)$ and $I_\nu(kz)$ behave as

$$K_\nu(kz) \to \mathrm{e}^{-kz}, \qquad I_\nu(kz) \to \mathrm{e}^{+kz}, \tag{3.40}$$

we impose the regularity condition at the IR, i.e. $B(k) = 0$.

Let us now consider the UV behavior $z \to 0$. In this region, we expand the Bessel function and observe

$$\begin{aligned} f_k(z) = A(k) z^2 K_\nu(kz) \to A(k) z^2 \times \frac{1}{2}\Gamma\big(\nu\big)\Gamma\big(1-\nu\big) \Bigg[&\left(\frac{kz}{2}\right)^{-\nu} \sum_{n=0} \frac{(kz/2)^{2n}}{n!\ \Gamma\big(n+1-\nu\big)} \\ &- \left(\frac{kz}{2}\right)^{\nu} \sum_{n=0} \frac{(kz/2)^{2n}}{n!\ \Gamma\big(n+1+\nu\big)} \Bigg] \\ \equiv A(k) z^2 \Big[z^{-\nu} \Big\{ a_0(k) + a_2(k) z^2 + \mathcal{O}(z^4) \Big\} &+ z^{\nu} \Big\{ b_0(k) + b_2(k) z^2 + \mathcal{O}(z^4) \Big\} \Big]. \end{aligned} \tag{3.41}$$

We could relate the overall coefficient $A(k)$ as the boundary value. We simply insert the solution to the expression (3.37) and fix the normalization through $\phi(x^\mu, z = \epsilon) \equiv \phi_{\mathrm{b}\epsilon}(x^\mu)$:

$$\phi(x^\mu, z) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\ \mathrm{e}^{ik_\mu x^\mu} A(k) z^2 K_\nu(kz) \quad \text{with} \quad A(k) = \frac{1}{\epsilon^2 K_\nu(k\epsilon)} \int \mathrm{d}^4 x\ \mathrm{e}^{-ik_\mu x^\mu} \phi_{\mathrm{b}\epsilon}(x^\mu). \tag{3.42}$$

The on-shell action can be evaluated by putting the solution (3.42) into the action (3.35),

$$\begin{aligned} S &= \frac{L^3}{2} \int \frac{\mathrm{d}^4 k\ \mathrm{d}^4 k'}{(2\pi)^4} \delta^4(k+k') \int_\epsilon^\infty \mathrm{d}z \frac{\mathrm{d}}{\mathrm{d}z} \Big(\frac{1}{z^3} f_k(z) f'_{k'}(z) \Big) \\ &= -\frac{L^3}{2} \int \frac{\mathrm{d}^4 k\ \mathrm{d}^4 k'}{(2\pi)^4} \delta^4(k+k') \int \mathrm{d}^4 x\ \mathrm{d}^4 x'\ \mathrm{e}^{-ik_\mu x^\mu} \mathrm{e}^{-ik'_\nu x'^\nu} \phi_{\mathrm{b}\epsilon}(x^\mu) \phi_{\mathrm{b}\epsilon}(x'^\mu) \left(\frac{1}{z^3} \frac{\big(z^2 K_\nu(kz)\big)'}{z^2 K_\nu(kz)} \right) \Bigg|_{z=\epsilon}, \end{aligned} \tag{3.43}$$

where we have used the equation of motion. The last piece in the on-shell action (3.43) can be estimated as[22]

$$\begin{aligned} \left(\frac{1}{z^3} \frac{\big(z^2 K_\nu(kz)\big)'}{z^2 K_\nu(kz)} \right) \Bigg|_{z=\epsilon} &= \frac{2+\nu}{\epsilon^4} - \frac{k}{\epsilon^3} \frac{K_{\nu+1}(\epsilon k)}{K_\nu(\epsilon k)} \\ &= \left\{ \frac{2-\nu}{\epsilon^4} + \mathcal{O}(1/\epsilon^2) \right\} + \left\{ 2\nu \underbrace{\left(-\frac{\Gamma\big(1-\nu\big)}{\Gamma\big(1+\nu\big)} \left(\frac{k}{2}\right)^{2\nu} \right)}_{= b_0(k)/a_0(k)} \epsilon^{2\nu-4} + \mathcal{O}(\epsilon^{2\nu-2}) \right\}, \end{aligned} \tag{3.44}$$

where $a_0(k)$ and $b_0(k)$ are given in the asymptotic expansion around $z \to \epsilon$ (3.41).

The first line of the expression (3.44) includes divergent terms. Since these coefficients are analytical functions of the 4D momentum k, we could remove them by adding appropriate local counter terms in

[22] We use the relations for the Bessel function $K_\nu(z)$:

$$K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z} K_\nu(z), \qquad K_{\nu-1}(z) + K_{\nu+1}(z) = -2K'_\nu(z).$$

the boundary without spoiling the equation of motion in the bulk [43]. The detail of this holographic renormalization program can be found in the review [44]. In the second line there exist non-analytic parts which could not be removed through counter terms. These terms contain physical information of 4D boundary theory.

We next perform the wave function renormalization

$$\phi_{\mathrm{b}\epsilon}(x^\mu)\epsilon^{\nu-2} \equiv \widetilde{\phi}_{\mathrm{b}}(x^\mu). \tag{3.45}$$

In this renormalization, the higher order terms in (3.44) vanish, such that we could access renormalized finite quantities. Then, we can obtain the two point function of the corresponding operator $\mathcal{O}(x)$ which couples the scalar source $\phi(x)$ via GKP-W relation (3.30),

$$\begin{aligned}\big\langle\mathcal{O}(x)\mathcal{O}(x')\big\rangle \equiv -\frac{\delta^2 S}{\delta\widetilde{\phi}_{\mathrm{b}}(x)\widetilde{\phi}_{\mathrm{b}}(x')} &= -2L^3\nu\frac{\Gamma\big(1-\nu\big)}{\Gamma\big(1+\nu\big)}\frac{1}{2^{2\nu}}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\ \mathrm{e}^{-ik(x-x')}k^{2\nu} \\ &= \frac{2L^3}{\pi^2}\nu\frac{\Gamma\big(\nu+2\big)}{\Gamma\big(\nu\big)}\frac{1}{|x-x'|^{2(2+\nu)}},\end{aligned} \tag{3.46}$$

where we have used the formula (2.157) to write the correlation function in real space. In the presence of the source term, the one-point function can be also estimated as

$$\big\langle\mathcal{O}(x)\big\rangle_{\widetilde{\phi}_{\mathrm{b}}} \equiv \frac{\delta S}{\delta\widetilde{\phi}_{\mathrm{b}}(x)} = -L^3\int\frac{\mathrm{d}^4k}{(2\pi)^4}\ 2\nu A(k)b_0(k)\ \mathrm{e}^{-ikx}. \tag{3.47}$$

Let us relate the result in terms of CFT. Near the boundary (3.41), we have

$$\phi(x^\mu,z) = \int\frac{\mathrm{d}^4x}{(2\pi)^4}\ \mathrm{e}^{ik_\mu x^\mu}A(k)\Big(z^{2-\nu}\big(a_0(k)+\mathcal{O}(z^2)\big)+z^{2+\nu}\big(b_0(k)+\mathcal{O}(z^2)\big)\Big). \tag{3.48}$$

We have identified the first part as the source term (3.45), while the second term as the VEV of the corresponding operator (3.47). Since we imposed the scale invariance of the CFT side in the relation (3.29), i.e. (3.33), we shall consider the scaling behavior in the gravity side. Requiring the scale invariance of the asymptotic solution (3.48) under the scaling $(x^\mu,z)\to(x'^\mu=\mathrm{e}^{-\lambda}x^\mu, z'=\mathrm{e}^{-\lambda}z)$ in 5D and (3.31) and (3.33), we find the condition

$$\Delta = 2+\nu = 2+\sqrt{4+L^2m^2}, \tag{3.49}$$

which relates the scaling dimension of 4D CFT to the mass of the 5D gravity theory. Comparing with the general argument from the CFT in (3.32), the two point function (3.46) satisfies the correct scaling behavior of the operator $\mathcal{O}(x^\mu)$ with the dimension $\Delta = 2+\nu$. For one-point function, if we send the source as $\widetilde{\phi}_0 = 0$, the one-point function (3.47) vanishes, since, in this case, the VEV is proportional to the source. This is consistent from the CFT point of view in which the VEV of the operator with non-zero scaling dimension should vanish. Nevertheless, the relation (3.47) plays the central role for the non-conformal application of the AdS/CFT correspondence [45, 46].

Let us reconsider what we did in this simple example and further read off some general structures. Of course what we need is to solve the equation of motion like (3.38), which is the second order ordinary differential equation. In order to tackle to such equations, it is standard to use the Frobenius series expansions. We can expand the solution locally around some point. For instance, around the boundary $z=0$, we set the solution as $f_k(z) = Cz^\alpha(1+\mathcal{O}(z))$ with a constant C and a exponent α. Inserting the solution into the equation of motion, we get the relation which the exponent α should satisfy. For the equation of motion (3.38), we get

$$0 = \alpha(\alpha-4) - L^2m^2, \quad \text{i.e.} \quad \alpha = 2\pm\sqrt{4+L^2m^2} = 2\pm\nu \equiv \Delta_\pm. \tag{3.50}$$

It is obviously understood that the asymptotic solution (3.48) is constructed in this way. The conformal dimension Δ (3.49) corresponds to the larger root of α i.e. Δ_+ in (3.50). These two modes with different exponents $\Delta_\pm$ could be the independent basis of the solution around the boundary:

$$\varphi(x^\mu, z) = \mathcal{A}(x^\mu)F^{(I)}(z) + \mathcal{B}(x^\mu)F^{(II)}(z), \tag{3.51}$$

with

$$F^{(I)}(z) = z^{\Delta_-}\big(1 + \mathcal{O}(z)\big) + c(z)F^{(II)}(z)\log z, \qquad F^{(II)}(z) = z^{\Delta_+}\big(1 + \mathcal{O}(z)\big),$$

where $\mathcal{A}(x^\mu)$ and $\mathcal{B}(x^\mu)$ are integration constants. Here the term proportional to $\log z$ may be needed to make the basis $F^{(I)}(z)$ and $F^{(II)}(z)$ linear independent, only in the case that the difference of two exponents is an integer. Coefficients in the expansion in the basis $F^{(I)}(z)$ and $F^{(II)}(z)$ including $c(z)$ could be obtained recursively[23]. In the similar way, for the massive scalar case, around the Killing horizon $z \to \infty$, we can find two basis (3.40) through the asymptotic form of the equation of motion $0 = f_k''(z) - k^2 f_k(z)$.

What we should do next is the "analytic continuation" to obtain the full solution via finding out the transformation matrix which could relate the two local basis in the different regions. For instance, the hypergeometric function may be constructed in this way with the help of the integral representations. In the most general cases, there exist no analytic expressions so that one needs to do these procedures by some numerical methods.

As we have done before, the integration constant $\mathcal{A}(x^\mu)$ could be identified as the source (c.f. (3.45)), while the other constant $\mathcal{B}(x^\mu)$ could be related to the VEV of the corresponding operators (c.f. (3.47)). Imposing the boundary condition in the interior of the AdS spacetime, $\mathcal{B}(x^\mu)$ and $\mathcal{A}(x^\mu)$ could be related, in general, nonlocally.

In order to complete the discussion, let us revisit the scaling dimension in more general ground. The scaling dimension is given by the root of the equation (3.50). In the CFT point of view, for scalar operators, there exists the bound $\Delta \geq 1$ by requiring the unitary representation. In the case $L^2m^2 \geq 0$, Δ_+ only satisfies this bound, and the dimension is given by $\Delta = \Delta_+ \geq 4$. Interesting thing happens in AdS spacetime. We can consider the negative mass satisfies the bound $L^2m^2 \geq -4$ which is known as Breitenlohner-Freedman bound [47]. Within this bound, the energy of the scalar field propagating the AdS spacetime can be positive due to the curvature effect of the geometry. When the mass decrees from 0 towards the BF bound, the root $\Delta_-(< 1$ for positive mass$^2)$ approaches to the unitary bound and may eventually satisfy the unitary bound in the case $-4 \leq L^2m^2 \leq -3$. Therefore, in this mass parameter region, we have two independent solutions which may correspond to two different boundary CFTs, i.e. $1 \leq \Delta_- \leq 2$ and $2 \leq \Delta_+ \leq 3$ [46].

Before closing this subsection, we display similar relations to (3.50) for lower spins cases

$$\text{vector} \qquad L^2m^2 = (\Delta - 1)(\Delta - 3) \tag{3.52a}$$

$$\text{symmetric tensor} \qquad L^2m^2 = \Delta(\Delta - 4) \tag{3.52b}$$

where m is the mass of the bulk field and Δ corresponds to the scaling dimension of the 4D operator. The massless 5D gauge field couples the dimension three operator which may correspond to a $U(1)$ conserved current in 4D CFT. The graviton couples to an operator with dimension four which would be the energy-momentum tensor.

4 AdS/CFT correspondence at finite temperature

After the AdS/CFT correspondence, many modifications of the original conjecture have been proposed and studied. We treat the AdS/CFT correspondence as a special case of the general bulk/boundary

[23]In general, the term proportional to $z^{\Delta_+ - \Delta_-}$ in $F^{(I)}(z)$ could not be fixed.

correspondence. One of the directions is to modify the gravity side in a suitable manner in order to study some features in the gauge theory side. One of the attempts is that for the thermal system [48].

4.1 Thermal field theory

Before embarking the application for AdS/CFT correspondence, let us briefly summarize the thermal field theory and the black hole thermodynamics.

We start by formulating the thermal field theories in equilibrium. The thermodynamic quantities could be determined by the canonical partition function at temperature T,

$$Z = \mathrm{Tr}\ \mathrm{e}^{-\frac{1}{T}H_0}, \tag{4.1}$$

where H_0 is the Hamiltonian.

For free bosonic/fermionic gas, we could compute the partition function (4.1) through the phase space integration,

$$\log Z = \mp V_3 \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \log\left(1 \mp \mathrm{e}^{-\frac{1}{T}\omega(\vec{p})}\right), \tag{4.2}$$

where $\mp$ correspond to boson and fermion, respectively, and $\omega(\vec{p})$ is the dispersion relation, i.e. $\omega(\vec{p}) = \sqrt{\vec{p}^2 + m^2}$ for relativistic particle. For later use, we display the results for the massless free particles for canonical ensemble:

$$\log Z = \begin{cases} V_3 \dfrac{\pi^2}{90} T^3 & \text{boson} \\ V_3 \dfrac{7\pi^2}{720} T^3 & \text{fermion} \end{cases} \tag{4.3}$$

We may evaluate the partition function (4.1) by Euclidean path-integral in the imaginary time formalism. We analytically continue to Euclidean spacetime and compactify the time direction with the thermal period $1/T$:

$$Z = \int \mathcal{D}\varphi\ \mathrm{e}^{-S_{\mathrm{E}}[\varphi]}, \quad \text{with} \quad S_{\mathrm{E}}[\varphi] = \int_0^{1/T} \mathrm{d}\tau \int \mathrm{d}\vec{x} \mathcal{L}(\varphi(\tau, \vec{x})), \tag{4.4}$$

where we need to impose periodic (anti-periodic) boundary conditions for bosonic (fermionic) fields $\varphi(\tau, \vec{x})$, which eventually break supersymmetry due to different types of the mode expansions around the period. The precise thermodynamic quantities can be found in the later sections.

4.2 Black hole thermodynamics

One could proceed with the gravitational system in the same spirit. In D-dimensional Euclidean spacetime, the evaluation of the partition function could be regarded as that of the canonical ensemble,

$$Z = \int \mathcal{D}g\ \mathrm{e}^{-S_{\mathrm{E}}[g]} = \sum_n\ \mathrm{e}^{-\frac{1}{T}E_n}, \tag{4.5}$$

where the action $S_{\mathrm{E}}[g]$ is typically Einstein-Hilbert term and E_n denotes the eigenvalue of the Hamiltonian.

In the semiclassical approximation, the most dominant contribution to the path-integral is from the solution of the classical equation of motion i.e. Einstein equation. Since the black hole is one of these solutions, the Einstein-Hilbert action evaluated at that configuration could be considered as the leading contributions to the free energy $F(= E - TS)$,

$$\mathrm{e}^{-\frac{1}{T}F} \equiv Z \sim \mathrm{e}^{-S_{\mathrm{E}}[g = g_{\text{black hole}}]}, \qquad \text{i.e.} \qquad F = T S_{\mathrm{E}}[g = g_{\text{black hole}}]. \tag{4.6}$$

As we will discuss, the temperature T could be defined as the Hawking temperature, which is a natural consequence from the black hole geometry. In order to evaluate the on-shell action $S_{\rm E}$, we need to regularize the action in general. The typical procedure is to add suitable counter terms to the pure gravitational action. By using the regularized on-shell action, we could obtain the thermal quantities, for instance the energy E and the entropy S as

$$E = -T^2 \frac{\partial S_{\rm E}}{\partial T}, \qquad S = \frac{E}{T} - S_{\rm E}. \tag{4.7}$$

If we consider the charged black hole so-called Reissner-Nordström black hole, we could access to the grand canonical partition function. The Reissner-Nordström black hole is the solution of the Einstein-Maxwell system,

$$S_{\rm E}[g, A] = S_{\rm EH}[g] + S_{\rm Maxwell}[g, A], \tag{4.8}$$

with

$$S_{\rm Maxwell}[g, A] = -\frac{1}{16\pi G_{\rm N}^{(D)}} \int_{\mathcal{M}} {\rm d}^D x \sqrt{g} F_{mn} F^{mn}. \tag{4.9}$$

The solution for the gauge potential is given by

$$A_{\rm RN} = A_t {\rm d}t = \Big(-\mu + \frac{Q}{r^{D-3}} \Big) {\rm d}t. \tag{4.10}$$

The integration constants μ and Q may correspond to the chemical potential and the number density, respectively. Through the semiclassical evaluation of the path-integral like (4.6), we equate the grand canonical potential $\Omega(\mu, T)$ to the on-shell action,

$$\Omega = T\Big(S_{\rm EH}[g = g_{\rm RN}] + S_{\rm maxwell}[g = g_{\rm RN}, A = A_{\rm RN}] \Big) = E - TS - \mu Q. \tag{4.11}$$

4.3 AdS-Schwarzschild black hole

We here combine two path-integral formulations discussed in the last two subsections in the context of the AdS/CFT correspondence. We extend the identification of the gauge/gravity theories in (3.26) for the thermal version. The bulk geometry in the gravity side can be replaced by certain black hole geometry. We equate the canonical partition function (4.6) for the gravity to that of the thermal gauge theory.

Let us be back to the initial place for the correspondence in the string theory framework. For our purpose, first we compactify 10D spacetime as $S^1 \times R^9$. In order to discuss a 4D world volume thermal gauge theory, we consider D3-branes wrapping on the Euclidean time circle S^1. In the context of AdS/CFT correspondence, we need to know the corresponding supergravity solutions. Here we use more general RR 3-brane solution (2.142)[24], which corresponds to non-BPS D3-brane with renaming the parameter $L \to \widetilde{L}$:

$$ {\rm d}s^2 = \frac{1}{\sqrt{1 + \frac{\widetilde{L}^4}{r^4}}} \Big\{ \Big(1 - \frac{r_{\rm H}^4}{r^4} \Big) {\rm d}\tau^2 + \sum_{i=1}^{3} {\rm d}x^{i2} \Big\} + \sqrt{1 + \frac{\widetilde{L}^4}{r^4}} \Big\{ \frac{{\rm d}r^2}{\Big(1 - \frac{r_{\rm H}^4}{r^4} \Big)} + r^2 {\rm d}\Omega_5^2 \Big\}, \tag{4.12}$$

where we work with the Euclidean time τ with the period $1/T$. In order to fix the integration constant $\widetilde{L}$, we use the identification (2.165) for the non-extremal solution. This identification gives the relation

$$\widetilde{L}^4 (\widetilde{L}^4 + r_{\rm H}^4) = L^8, \tag{4.13}$$

[24]One can consider Euclidean version of the extremal solution (2.143) with S^1 compactification. However, by comparing the free energies, one could see that the solution (2.142) is energetically favored.

where L is the same as (3.7). Taking the limit $r_{\mathrm{H}} \to 0$, the solution is obviously reduced to the extremal D3-brane solution.

The metric (4.12) contains a coordinate singularity at $r = r_{\mathrm{H}}$ which represents the event horizon. Let us look at the horizon closely,

$$\mathrm{d}s^2|_{r\to r_{\mathrm{H}}} = \frac{4r_{\mathrm{H}}}{\sqrt{r_{\mathrm{H}}^4+\widetilde{L}^4}}(r-r_{\mathrm{H}})\mathrm{d}\tau^2 + \frac{\sqrt{r_{\mathrm{H}}^4+\widetilde{L}^4}}{4r_{\mathrm{H}}(r-r_{\mathrm{H}})}\mathrm{d}r^2 + \cdots = \frac{\sqrt{r_{\mathrm{H}}^4+\widetilde{L}^4}}{r_{\mathrm{H}}^2}\Big(\mathrm{d}\xi^2 + \frac{4r_{\mathrm{H}}^2}{r_{\mathrm{H}}^4+\widetilde{L}^4}\xi^2\mathrm{d}\tau^2\Big) + \cdots, \tag{4.14}$$

where we have introduced the coordinate ξ as the deviation from the horizon $r = r_{\mathrm{H}} + \xi^2/r_{\mathrm{H}}$. We can regard the Euclidean time coordinate τ as the angular variable $\theta \equiv \big(2r_{\mathrm{H}}/\sqrt{r_{\mathrm{H}}^4+\widetilde{L}^4}\big)\tau$. Imposing the periodicity at $\xi = 0$, i.e. removing the conical singularity, we obtain

$$2\pi = \frac{2r_{\mathrm{H}}}{\sqrt{r_{\mathrm{H}}^4+\widetilde{L}^4}}\frac{1}{T}, \qquad \text{i.e.} \qquad T = \frac{r_{\mathrm{H}}}{\pi\sqrt{r_{\mathrm{H}}^4+\widetilde{L}^4}}. \tag{4.15}$$

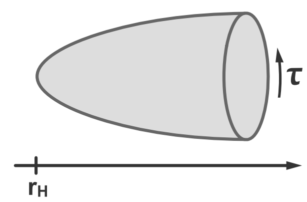

Figure 12: τ-r surface around the horizon

This is the Hawking temperature which could be identified with that of the gauge theory. The previous D3-brane solution which is realized by $r_{\mathrm{H}} = 0$ corresponds to zero temperature, i.e. the ground state.

Let us now consider the decoupling limit. This field theory limit may be taken to be $\alpha' \to 0$ with keeping the temperature fixed in addition to the gauge coupling constant. By using the relations (4.13), (4.15), and $L \propto \sqrt{\alpha'}$, we can estimate the scaling behaviors as $\widetilde{L} \propto \sqrt{\alpha'}$ and $r_{\mathrm{H}} \propto \alpha'$. Eliminating r_{H} from the relations (4.13) and (4.15), we could obtain the deviation from extremality,

$$\widetilde{L}^4 = L^4\Big(1 - \frac{1}{2}(\pi^4L^4T^4) + \mathcal{O}(L^8T^8)\Big). \tag{4.16}$$

As we did in the D3-brane case, keeping the energy scale $u = r/L^2$ i.e. $r \propto \alpha'$, we obtain the metric (4.12) as

$$\frac{\mathrm{d}s^2}{L^2} = u^2\bigg(\Big(1-\frac{u_{\mathrm{H}}^4}{u^4}\Big)\mathrm{d}\tau^2 + \sum_{i=1}^{3}\mathrm{d}x^{i2}\bigg) + \frac{1}{u^2}\frac{\mathrm{d}u^2}{\Big(1-\frac{u_{\mathrm{H}}^4}{u^4}\Big)} + \mathrm{d}\Omega_5^2, \tag{4.17}$$

where we have denoted $u_{\mathrm{H}} = r_{\mathrm{H}}/L^2$ which is order one in the decoupling limit. This near horizon geometry is (5D AdS-Schwarzschild black hole)$\times S^5$ with a common radius L, which provides the dual description of the boundary gauge theory at finite temperature.

We can calculate thermodynamic quantities and verify the thermodynamic relations. In the near horizon limit, the mass (energy) can be calculated from (2.139) by using the relation (4.16),

$$\frac{M}{V_3} = T_3N_{\mathrm{c}} + \frac{3}{8}\pi^2N_{\mathrm{c}}^2T^4, \tag{4.18}$$

where in the near horizon limit, the temperature (4.15) is given by

$$T = \frac{r_{\mathrm{H}}}{\pi L^2}. \tag{4.19}$$

The first term is from the N_{c} D3-branes tension and the second term corresponds to the energy density of the field theory. The entropy can be also calculated as the Bekenstein-Hawking entropy in the context of the black hole thermodynamics,

$$S = \frac{(\text{Area})}{4G_{\mathrm{N}}^{(D)}}. \tag{4.20}$$

At the horizon, the metric (4.17) has the form $\mathrm{d}s^2|_{u=u_{\mathrm{H}}} = L^2 u_{\mathrm{H}}^2 \sum_{i=1}^{3} \mathrm{d}x^{i2} + L^2 \mathrm{d}\Omega_5^2$. Therefore, the area can be computed as Area $= \sqrt{(L^2 u_{\mathrm{H}}^2)^3 L^{10}}\ V_3 V_{S^5}$, and we obtain the entropy density

$$s = \frac{S}{V_3} = \frac{\pi^2}{2} N_{\mathrm{c}}^2 T^3. \tag{4.21}$$

These energy and entropy could be also reproduced through the evaluation of the on-shell action (4.7) with identifying the free energy of the black hole with that of the gauge theory.

Let us compare the result with that of the free gas approximation. The physical on-shell degrees of freedom for the field theory i.e. $SU(N_{\mathrm{c}})$ $\mathcal{N} = 4$ super Yang-Mills which we have discussed are $N_{\mathrm{c}}^2 \times \big(2 \text{ (gauge field)} + 6 \text{ (scalar)}\big)$ for the bosons and $N_{\mathrm{c}}^2 \times 8$ $(= N_{\mathrm{c}}^2 \times 2 \times 4)$ for 4 Weyl fermions. Therefore, summing up the degrees of freedom for (4.3), we find

$$\log Z = \frac{\pi^2}{6} V_3 N_{\mathrm{c}}^2 T^3. \tag{4.22}$$

Since the entropy can be calculated via $S = \partial(T \log Z)/\partial T$, we then conclude [49]

$$s_{\text{black hole}} = \frac{3}{4} s_{\text{free gas}}. \tag{4.23}$$

The entropy for the strong coupling regime may be reduced from the free system by the factor 3/4.

In the results (4.18) and (4.21), the scaling T^3 could be understood by the conformal invariance. The factor N_{c}^2 indicates the theory is in the deconfined phase where the color degrees of freedom could be visible.

4.4 Finite density

The AdS/CFT correspondence can be also extended to the case with a finite charge density. A $U(1)$ gauge symmetry in the AdS black hole background may be dual to a global $U(1)$ symmetry in the gauge theory side. Introducing the gauge field to the gravity side, we could consider the grand potential (4.11) and discuss the thermodynamics and related phase structure.

If we consider the gauge field in the bulk gravity theory, the equation (3.26) may be the form,

$$\Big\langle\ \mathrm{e}^{\int \mathrm{d}^4x A_\mu(x^m) J^\mu(x^\mu)}\ \Big\rangle_{\mathrm{CFT}} = \mathrm{e}^{-S_{\text{supergravity}}[A_\mu(x^\mu)]}\Big|_{\text{on-shell}} \quad \text{with} \quad A_\mu(x^\mu) = A_\mu(x^M)\big|_{\text{boundary}}. \tag{4.24}$$

In the left hand side in (4.24), the boundary value of the bulk gauge field $A_\mu(x^\mu)$ couples to the current operator $J^\mu(x^\mu)$ which should be a conserved current due to the gauge invariance of the coupling. The boundary value of the time component of the gauge field $A_t(x^\mu)$ could be identified with the chemical potential μ associated with the density in the gauge theory [25],

$$\mu = A_t(x^M)|_{\text{boundary}}. \tag{4.25}$$

This could be understood in the following way. Under the interpretation (4.25), the left hand side of the equation (4.24) may become the grand canonical partition function of the gauge theory

$$\Big\langle\ \mathrm{e}^{\int \mathrm{d}^4x A_\mu(x^m) J^\mu(x^\mu)}\ \Big\rangle_{\mathrm{CFT}} = \Big\langle\ \mathrm{e}^{\int_0^{1/T} \mathrm{d}\tau \mu Q}\ \Big\rangle_{\mathrm{CFT}} = \int \mathcal{D}\phi\ \mathrm{e}^{-\int_0^{1/T} \mathrm{d}\tau (H_0 - \mu Q)}, \tag{4.26}$$

where the charge Q is defined by usual way $Q = \int \mathrm{d}^3x J^t(x)$.

[25] In the presence of the black hole, one could define the chemical potential in the gauge invariant way, $\mu = \int_{r_{\mathrm{H}}}^{\infty} \mathrm{d}r F_{rt} = A_t(\infty) - A_t(r_{\mathrm{H}}) = A_t(\infty)$, where we have used the natural condition for the gauge potential at the horizon $A_t(r_{\mathrm{H}}) = 0$.

In the context of the AdS/CFT correspondence, the $U(1)$ gauge symmetry can be naturally introduced [50]. In the static non-extremal black 3-brane solution (4.12), we turn on three different angular momenta in the 6D transverse space to the world volume. Specifying the three rotation axes breaks the $SO(6)$ rotational symmetry to Cartan parts $U(1)^3$. As in the usual AdS/CFT correspondence, we next take the near horizon limit with keeping the rotational parameters constant. Then, the rotating 3-brane metric becomes a product spacetime consisting of asymptotic AdS_5 and modified S^5. Identifying the rotational parameters with charges, the asymptotic AdS_5 part could be understood as the black hole with three charges, known as the STU AdS black hole [51]. This is the solution of the 5D $\mathcal{N}=2$ gauged supergravity which comes from the consistent truncation of the $SO(6)$ gauged $\mathcal{N}=8$ supergravity. As we mentioned, the $SO(6)$ gauged $\mathcal{N}=8$ supergravity is considered as a consistent S^5 reduction of type IIB supergravity. If the three charges are the same, the solution is simply reduced to 5D Reissner-Nordström-AdS black hole which is the solution of equations of motion given by the action (4.8) with a negative cosmological constant. Applying the AdS/CFT correspondence to this simple model, one could discuss thermal physics especially in the hydrodynamic limit [52].

5 D3/D7 model

We have discussed the gauge/gravity duality and extended the idea to describe the gauge theory with finite temperature and density. In the gauge theory, so far, we have only considered the adjoint fields. Two endpoints of open strings which correspond to point charges in the fundamental/anti-fundamental representations have been attached to the same branes, so that they transform in the adjoint representation of the $SU(N_\mathrm{c})$ gauge group. In order to consider the fundamental matter with flavors, we need to introduce open strings whose two endpoints, both of them, are not attached to the color branes [53]. This can be realized by introducing different types of D-branes. Since the fundamental matter lives in 4D, this 4D spacetime should be filled by these Dp-branes ($p>3$). In this section we consider D3/D7 model which describes $SU(N_\mathrm{c})$ $\mathcal{N}=4$ super Yang-Mills with $\mathcal{N}=2$ N_f quark hypermultiplets.

We consider a stack of N_c D3-branes and another stack of N_f D7-branes. The D-brane configuration is given by Table 1 and several open strings attached on the D-branes are visualized in Fig.13. As we

	t	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D3	•	•	•	•						
D7	•	•	•	•	•	•	•	•		

Table 1: The brane configurations: the background D3- and the probe D7-branes

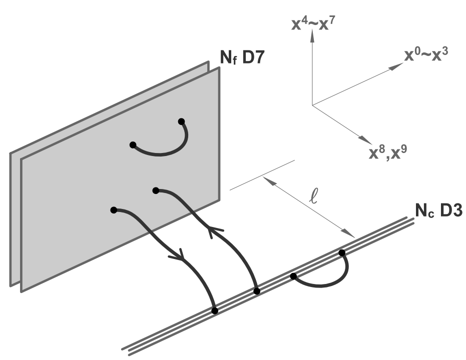

Figure 13: D3/D7 configuration.

have discussed, we could see $\mathcal{N}=4$ super Yang-Mills sector coming from open strings attached in the

D3-branes. Quarks are described by the open strings stretched between D3- and D7-branes. This is the case with $n = 4$ (number of (ND) directions) in (2.126a) and (2.126b). The R sector provides the fermion with mass $M_{\rm q} = l/(2\pi\alpha')$, while the NS sector provide the scalar with the same mass. This indicates that the model has supersymmetry. There also exist open string modes in D7-branes which provide the $U(N_{\rm f})$ gauge theory in their world volume. These adjoint representations are naturally considered as the mesonic degrees of freedom. These states are described by the fluctuation of the D7-branes.

Now we shall discuss the decoupling limit $\alpha' \to 0$. From (2.151), the gauge coupling constants have different α' scalings for D3- and D7-branes,

$$g^2_{\rm YM\ D7} = (2\pi)^4\alpha'^2 g^2_{\rm YM}, \tag{5.1}$$

where $g^2_{\rm YM}$ is the D3-brane Yang-Mills coupling constant given by (3.3). Taking the limit $\alpha' \to 0$ while keeping the 4D Yang-Mills coupling constant, $g^2_{\rm YM\ D7}$ vanishes. Therefore, the $U(N_{\rm f})$ gauge symmetry on $N_{\rm f}$ D7-branes becomes global one. Moreover, since the interaction terms between strings (D3-D7) and those for (D7-D7) are proportional to $g^2_{\rm YM\ D7}$, the (D7-D7) sector could be decoupled to the other sectors (D3-D3) and (D3-D7). As a result, bifundamental fields which come from (D3-D7)/(D7-D3) strings transform as the fundamental/antifundamental representations of the $SU(N_{\rm c})$ gauge group and $U(N_{\rm f})$ global flavor group.

In order to discuss the gauge theory in the context of the AdS/CFT correspondence, we need to have a dual gravity picture. In general, it is hard to obtain supergravity solutions for these coupled D-brane system. Then one can use so-called the probe approximation [4, 6, 53]. We here would like to treat D3-branes as the "background" and D7-branes as the probe to this background. Just like the quenched approximation in lattice QCD in studying the effect of a test quark in the given gluon background, we assume that the probe flavor D7-branes do not affect the geometry. This could be achieved by $N_{\rm c} \gg N_{\rm f}$ such that the gravitational backreaction could be suppressed by $N_{\rm f}/N_{\rm c}$. This could be easily observed in the Newton potential as in (2.169),

$$8\pi G_{\rm N}^{(10)} T_{tt}\Big|_{\rm D7} \propto \lambda_{\rm D7} = N_{\rm f} g_{\rm s} = \lambda \frac{N_{\rm f}}{N_{\rm c}}.$$

Therefore, we could separately solve the equations of motion for the gravity sector. As a result, D3-branes provide $\mathrm{AdS}_5 \times S^5$ background geometry, and in this background we consider the consistent embedding of the D7-branes, which is determined by the DBI action (2.146).

5.1 Thermodynamics

In this section, we briefly summarize D3/D7 system at finite temperature. Following [54, 55], we consider the "black hole embedding" of the probe D7-branes in the D3-brane background in which we could introduce the finite quark mass together with the chemical potential.

We start by introducing non BPS N_c D3-branes. The gravity dual descriptions is given by the metric (4.17);

$$\mathrm{d}s^2 = \frac{u^2}{L^2}\Big(f_0(u)\mathrm{d}t^2 + \sum_{i=1}^{3}\mathrm{d}x^{i2}\Big) + \frac{L^2}{u^2}\Big(\frac{\mathrm{d}u^2}{f_0(u)} + u^2\mathrm{d}\Omega_5^2\Big), \qquad \text{with} \qquad f_0(u) = 1 - \frac{u_{\rm H}^4}{u^4}, \tag{5.2}$$

where we have rescaled the radial coordinate as $u \to u/L^2$ in the expression (4.17). In the D3-brane background, the background dilaton is constant $\mathrm{e}^{\widetilde{\Phi}} = 1$ in our notation. Following the paper [4,55], we introduce new coordinate ϱ through $\varrho^2 = u^2 + \sqrt{u^4 - u_{\rm H}^4}$, and rewrite the metric (5.2) as

$$\mathrm{d}s^2 = \frac{1}{2}\frac{\varrho^2}{L^2}\left(\frac{f^2(\varrho)}{\tilde{f}(\varrho)}\mathrm{d}t^2 + \tilde{f}(\varrho)\sum_{i=1}^{3}\mathrm{d}x^{i2}\right) + \frac{L^2}{\varrho^2}\Big(\mathrm{d}\varrho^2 + \varrho^2\mathrm{d}\Omega_5^2\Big), \text{ with } f(\varrho) = 1 - \frac{u_{\rm H}^4}{\varrho^4},\ \tilde{f}(\varrho) = 1 + \frac{u_{\rm H}^4}{\varrho^4}. \tag{5.3}$$

We work with the dimensionless coordinate $\rho \equiv \varrho/u_{\rm H}$. We refer to the horizon as $\rho = 1$ and the AdS boundary as $\rho\ (= \sqrt{2}u/u_{\rm H}) \to \infty$.

Next, we introduce $N_{\rm f}$ D7-branes as the flavor branes with the intersection given in Fig.13 where D7-branes are wrapping on S^3 of S^5. Taking the probe approximation $N_{\rm f} \ll N_{\rm c}$, we could consider the dynamics of $N_{\rm f}$ D7-branes which can be described by the DBI action (2.146) in the 10D background (5.3),

$$S_{\rm D7} = N_f T_7 \int {\rm d}^8\sigma \sqrt{\det(g_{mn} + 2\pi\alpha' F_{mn})}, \tag{5.4}$$

where we do not consider the fluctuations of the dilaton $\widehat{\Phi}(\sigma)$ and the Kalb-Ramond field $B_{mn}(\sigma)$. It is convenient to divide the transverse 6D part to the D3-branes in (5.3) into two parts i.e. 4D and 2D whose coordinates are given by spherical (r, Ω_3) and polar (R, φ) coordinates, respectively,

$$\begin{aligned}({\rm d}\rho)^2 + \rho^2 {\rm d}\Omega_5^2 &= ({\rm d}r)^2 + r^2 {\rm d}\Omega_3^2 + ({\rm d}R)^2 + R^2({\rm d}\varphi)^2 \\ &= ({\rm d}\rho)^2 + \rho^2\Big(({\rm d}\theta)^2 + \sin^2\theta\, {\rm d}\Omega_3^2 + \cos^2\theta\, ({\rm d}\varphi)^2\Big),\end{aligned} \tag{5.5}$$

where $r = \rho\sin\theta$, $R = \rho\cos\theta$ with $0 \le \theta \le \pi/2$ and $\rho^2 = r^2 + R^2$. By construction, we consider the case where the world volume coordinates of D7-branes are given in the static gauge i.e. $\sigma^m \equiv (t, x^i, \rho, \Omega_3)$. Due to the symmetries for the translation in (t, x^i) and the rotation in (ρ, Ω_3), the embedding of the D7-branes could depend only on the radial coordinate ρ. Since the rotational symmetry in (R, φ) allows to set $\varphi = 0$, the embedding might be characterized by $\chi(\rho) \equiv \cos\theta$ through $\theta(\rho)$ which is the angle between two spaces (r, Ω_3) and (R, φ). The asymptotic value of the distance between D3 and D7-branes which is given by $R(\rho)$ for large ρ provides the quark mass $M_{\rm q}$.

The induced metric on the D7-branes can be obtained as

$${\rm d}s^2_{\rm D7} = L^2\left\{\frac{\pi^2T^2}{2}\rho^2\left(\frac{f^2}{\tilde f}{\rm d}t^2 + \tilde f\sum_{i=1}^{3}{\rm d}x^{i2}\right) + \frac{1}{\rho^2}\left(\frac{1-\chi^2+\rho^2\dot\chi^2}{1-\chi^2}\right){\rm d}\rho^2 + \left(1-\chi^2\right){\rm d}\Omega_3^2\right\}, \tag{5.6}$$

where the dot stands for the derivative with respective to ρ. We also introduce the non-dynamical temporal component of the gauge field $A_t(\rho)$ which incorporates the chemical potential and the density at the AdS boundary.

By using the induced metric (5.6) and the gauge potential $A_t(\rho)$, the DBI action (5.4) now becomes

$$S_{\rm D7} = \frac{\lambda N_c N_f T^3}{32} V_3 \int {\rm d}\rho\ \rho^3 \tilde f(1-\chi^2)\sqrt{f^2(1-\chi^2+\rho^2\dot\chi^2) - 2\tilde f(1-\chi^2)\dot{\widetilde{A}}_t^2}, \tag{5.7}$$

where we have defined $\widetilde{A}_t(\rho) \equiv 2\pi\alpha' A_t(\rho)/u_{\rm H}$ and the 't Hooft coupling $\lambda = g_{\rm YM}^2 N_{\rm c}$. Since there exist no $\widetilde{A}_t(\rho)$ terms in the action, the equation of motion for $\widetilde{A}_t(\rho)$ can be reduced to the following form with an integration constant $\tilde d$,

$$\tilde d \equiv \frac{\rho^3 \tilde f^2 (1-\chi^2)^2 \dot{\widetilde{A}}_t}{2\sqrt{f^2(1-\chi^2+\rho^2\dot\chi^2) - 2\tilde f(1-\chi^2)\dot{\widetilde{A}}_t^2}}. \tag{5.8}$$

The equation of motion for $\chi(\rho)$ is given as

$$\begin{aligned}0 = \partial_\rho&\left\{\frac{\rho^5 f\tilde f(1-\chi^2)\dot\chi}{\sqrt{1-\chi^2+\rho^2\dot\chi^2}}\left(1+\frac{8\tilde d^2}{\rho^6\tilde f^3(1-\chi^2)^3}\right)^{1/2}\right\} \\ &+\frac{\rho^3 f\tilde f\chi}{\sqrt{1-\chi^2+\rho^2\dot\chi^2}}\left\{\Big(3(1-\chi^2)+2\rho^2\dot\chi^2\Big)\left(1+\frac{8\tilde d^2}{\rho^6\tilde f^3(1-\chi^2)^3}\right)^{1/2}\right. \\ &\qquad\left. - \frac{24\tilde d^2(1-\chi^2+\rho^2\dot\chi^2)}{\rho^6\tilde f^3(1-\chi^2)^3}\left(1+\frac{8\tilde d^2}{\rho^6\tilde f^3(1-\chi^2)^3}\right)^{-1/2}\right\},\end{aligned} \tag{5.9}$$

where we have eliminated the gauge field $\widetilde{A}_t(\rho)$ by using the relation (5.8). Near the boundary, asymptotic solutions of the equations of motion (5.8) and (5.9) behave as

$$\widetilde{A}_t(\rho) = \widetilde{\mu} - \frac{\widetilde{d}}{\rho^2} + \cdots, \qquad \chi(\rho) = \frac{m}{\rho} + \frac{c}{\rho^3} + \cdots. \tag{5.10}$$

The chemical potential μ can be defined as the boundary value of $A_t(\rho)$, while the quark mass $M_{\rm q}$ can be estimated through the asymptotic value of the separation of D3 and D7-branes i.e. $M_{\rm q} = \rho\chi(\rho)/(2\pi\alpha')$ at the AdS boundary. Taking the rescaling of the gauge field $\widetilde{A}_t$ and the coordinate ρ into account, the integration constants $\widetilde{\mu}$ and m can be related to these values,

$$\widetilde{\mu} = \frac{2\pi\alpha'}{u_0}\mu = \sqrt{\frac{2}{\lambda}}\frac{\mu}{T}, \qquad m = 2\pi\alpha'\frac{\sqrt{2}}{u_0}M_{\rm q} = \frac{2}{\sqrt{\lambda}}\frac{M_{\rm q}}{T}. \tag{5.11}$$

As we will estimate below[26], the remaining constants $\widetilde{d}$ and c would be proportional to the quark number density $n_{\rm q}$ and the quark condensate $\langle\bar{\psi}\psi\rangle$, respectively.

It is easy to observe that the free part of a linearized equation of motion for the scalar field $\chi(\rho)$ (5.9), i.e. $0 = \partial_\rho(\rho^5 f\tilde{f}\dot{\chi}) + 3\rho^3 f\tilde{f}\chi$ is equivalent to the Klein-Gordon equation in the AdS-Schwarzschild background (5.3) with mass $m^2L^2 = -3$. By using the scaling argument for the scalar field (3.50)[27], it is confirmed that the scaling dimension of the corresponding operator in the boundary theory may be three. The similar argument could be confirmed for the massless gauge field. The asymptotic solution of $\widetilde{A}_t(\rho)$ in (5.10) supports the scaling (3.52a) for a conserved current in 4D boundary.

In order to solve the nonlinear equations of motion (5.8) and (5.9), we need to use numerical methods. Here we restrict to the black hole embedding in which the D7-branes touch the horizon since this might be thermodynamically favored configuration in the system with finite density. We impose boundary conditions at the horizon as $\dot{\chi}(1) = 0$, $\widetilde{A}_t(1) = 0$ to remove singularities and $\chi(1) = \chi_0$. We fix m and $\widetilde{\mu}$ which depend on χ_0 and $\widetilde{d}$ by matching the numerical solutions with the asymptotic forms at the boundary.

We now consider the on-shell action which is related to the partition function Z of the field theory in the context of AdS/CFT correspondence. However the on-shell action contains UV divergences. It is well-known that one can prepare local boundary counter terms for probe D-branes in AdS spacetime by applying the holographic renormalization [56]. In the D3/D7 system, taking the asymptotic solution (5.10) into account, the relevant boundary counter terms take the form [57,58],

$$S_{\rm ct} = \frac{\lambda N_c N_f T^3}{32}V_3\left\{ -\frac{1}{4}\Big((\rho_{\rm max}^2 - m^2)^2 - 4mc\Big)\right\}, \tag{5.12}$$

where $\rho_{\rm max}$ is the cut-off for UV divergences which may go to infinity after precise calculations. It should be noticed that there exist finite contributions in the counter terms. Together with these counter terms, we could obtain the regularized action

$$\begin{aligned} S_{\rm D7\ reg} &= S_{\rm D7} + S_{\rm ct} \\ &= \frac{\lambda N_c N_f T^3}{32}V_3\left\{ \int_1^\infty {\rm d}\rho\left(\rho^3\tilde{f}(1-\chi^2)\sqrt{f^2(1-\chi^2+\rho^2\dot{\chi}^2) - 2\tilde{f}(1-\chi^2)\dot{\widetilde{A}}_t^2}\right.\right. \\ &\qquad\qquad \left.\left. -\rho^3 + m^2\rho\right) - \frac{1}{4}\Big((m^2-1)^2 - 4mc\Big)\right\}. \end{aligned} \tag{5.13}$$

[26]For the dimension three operator, we neglect contributions from squarks in the hypermultiplet.

[27]Strictly speaking, we should move on to the extremal case where we can define the boundary CFT.

Since we are interested in the black hole embedding, the integration supports from the horizon to the AdS boundary. Evaluating the on-shell action for (5.13) which is reduced to boundary values through the equation of motion, we could observe that the quark condensate is proportional to the integration constant c,

$$\langle\bar{\psi}\psi\rangle = -\frac{T}{V_3}\frac{\partial}{\partial M_{\mathrm{q}}}\log Z = -\frac{1}{8}\sqrt{\lambda}N_{\mathrm{c}}N_{\mathrm{f}}T^3 c, \tag{5.14}$$

where we have used the asymptotic solutions (5.10) and the relation (5.11). Since we identify the grand potential as $\Omega = -T\log Z$, the quark number density can be also calculated through the on-shell evaluation,

$$n_{\mathrm{q}} = -\frac{1}{V_3}\frac{\partial\Omega}{\partial\mu} = \frac{1}{4}\sqrt{\frac{\lambda}{2}}N_{\mathrm{c}}N_{\mathrm{f}}T^3\widetilde{d}. \tag{5.15}$$

5.2 Parton energy loss and quarkonium dissociation

Now, we discuss a few probes of the QGP in this D3/D7 model; Some general discussion on the QGP can be found in [59–62]. We will consider the energy loss of partons and dissociation of the (heavy) quarkonium.

The energy loss of partons is a useful probe of the QGP. Though the energy loss is not a direct experimental observable, it is manifested in jet quenching, a signature of the QGP. Jet quenching in AdS/CFT was initially discussed in [63, 64]. Traveling through the dense medium, energetic partons will lose their energy. There are two primary sources of the energy loss: collisional and radiative. The collisional energy loss is due to the scattering of the energetic partons with thermal quarks and gluons in the QGP, see Fig.14 for a sample process, while the radiative one is attributed to the Bremsstrahlung during the interactions with the medium, see Fig.15 for a typical diagram. We refer to [65, 66] for a review on the physics of jet quenching.

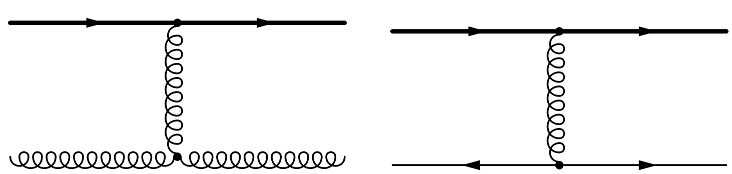

Figure 14: Typical diagrams for the collisional energy loss. Thick solid line is for energetic quarks traveling through medium, interacting with thermal quarks (thin solid lines) and gluons (helical lines).

Now we demonstrate how one can calculate the parton energy loss in the D3/D7 model at finite temperature discussed above following [67, 68]. The energy loss per unit length of a quark moving through the medium with velocity v obtained in [67] is given by

$$\frac{\mathrm{d}E}{\mathrm{d}x} = \frac{1}{v}\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{\pi}{2}\sqrt{\lambda}T^2\frac{v}{\sqrt{1-v^2}}. \tag{5.16}$$

Below, we will show how this result comes out.

$\mathcal{N}=4$ $SU(N_{\mathrm{c}})$ super Yang-Mills theory with finite temperature has its gravity dual description (5.2). The probe D7-brane wraps on S^3 of S^5 and fills all of the asymptotically AdS$_5$ down to a minimum radial value u_m. To calculate the energy loss, we consider a classical open string configuration whose one endpoint attached on the D7-brane and another one stretches to the horizon of black hole, see Fig.16. This string corresponds to a quark whose mass is related to the minimum radius u_m. The dynamics of this string can be described by the Nambu-Goto action (2.65)

$$S_{\mathrm{NG}} = -\frac{1}{2\pi\alpha'}\int \mathrm{d}\tau\mathrm{d}\sigma\,\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}, \tag{5.17}$$

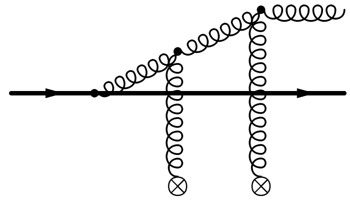

Figure 15: A typical process for the radiative energy loss. Thick solid line denotes fast moving quarks through the QGP. $\otimes$ is for colored static scattering sources in the medium.

where the dot and the prime stand for a derivative with respect to τ and σ, respectively. Without loss of generality, we can assume that the string lives on three-dimensional slice of an asymptotically AdS_5. The three-dimensional slice is described by the coordinates (t, u, x), where x is one of the transverse coordinates x^i. With the choice of a static gauge $t = \tau$ and $u = \sigma$, the shape of the string is described by a single variable, $x = x(t, u)$. Then, the determinant of the induced metric g_{mn} becomes

$$g \equiv \det g_{mn} = -\Big(1 - \frac{\dot{x}^2}{f(u)} + \frac{u^4 f(u)}{L^4} x'^2\Big). \tag{5.18}$$

The equation of motion for the string configuration reads

$$0 = \partial_u \left(\frac{u^4 f(u) x'}{\sqrt{-g}} \right) - \frac{L^4}{f(u)} \partial_t \left(\frac{\dot{x}}{\sqrt{-g}} \right). \tag{5.19}$$

For simplicity, we will consider the string moving with constant velocity v, which corresponds to quarks moving under a constant electric field in gauge theory side.

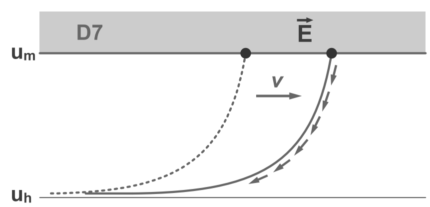

Figure 16: A classical string configuration. The downwards arrow indicates the energy (momentum) flow along the string.

With the ansatz $x(t, u) = x(u) + vt$ and the condition that the action should be real, one can obtain the solution of (5.19) as

$$x(t, u) = x_0 + \frac{L^2 v}{2u_{\rm H}} \left(\frac{\pi}{2} - \tan^{-1} \left(\frac{u}{u_{\rm H}} \right) - \coth^{-1} \left(\frac{u}{u_{\rm H}} \right) \right) + vt. \tag{5.20}$$

With this solution one can easily calculate the energy (or momentum) flow along the string which is given by the canonical momentum densities

$$\pi^\sigma_M \equiv \frac{\partial \mathcal{L}}{\partial X'^M} = -\frac{G_{MN}}{2\pi\alpha'} \frac{(\dot{X} \cdot X')\dot{X}^N - \dot{X}^2 X'^N}{\sqrt{-g}}, \tag{5.21}$$

where $\mathcal{L}$ is the Lagrangian of the Nambu-Goto action. If we consider $M = t$ case, we obtain the energy flow along the string to the black hole horizon, or minus of the energy loss of the quark,

$$\frac{{\rm d}E}{{\rm d}t} = \frac{\pi}{2} \sqrt{\lambda} T^2 \frac{v^2}{\sqrt{1 - v^2}}. \tag{5.22}$$

Based on the idea that off-shell gluons in a thermal medium can be viewed as a string with its both endpoints passing through the horizon of an AdS black hole, the energy loss of an energetic gluon in a thermal plasma was studied in [69]. An interesting idea of heavy quark energy loss through the Cherenkov radiation of mesons by quarks was explored in the gauge/gravity duality [70]. For more on energy loss including recent developments, we refer to [71, 72] and references therein.

Now, we move onto another probe of the QGP. The thermal dissociation of the (heavy) quark bound states, which might be due to Debye screening in the QGP, will convey signals of onset of new state of matter (QGP). There are a few ways to study heavy quarkonium properties at high temperature. An obvious and simple way is to consider linearized equations of motion for fluctuations of the D-brane embeddings. In this case, however, one has to be cautious about the fact that the bound states in Dp/Dq configurations are deeply bound, while heavy quarkonia are shallow bound states [58]. Holographic meson melting through the quasinormal modes of D7-brane fluctuations is studied in [73]. One can also solve Schrödinger equation with temperature-dependent interquark potentials derived from a holographic model [74] or calculate the holographic finite-temperature spectral function of (heavy) quarkonia [75, 76].

6 D4/D8/$\overline{\text{D8}}$ model

One of the major goals in the gauge/gravity duality is to find holographic models which capture features of QCD, in other words, to find a dual geometry for QCD. Witten proposed a construction of the holographic dual of 4D pure $SU(N_{\rm c})$ Yang-Mills theory [48]. The confinement/deconfinement could be discussed in geometrical manner. In order to introduce the flavors, D8/$\overline{\text{D8}}$-branes would be considered in this section. In addition to the confinement/deconfinement, this D4/D8/$\overline{\text{D8}}$ model so-called Sakai-Sugimoto model [7] nicely describes the nonabelian chiral symmetries. In this section, we briefly sketch the D4-brane background and D8/$\overline{\text{D8}}$ flavor branes. We yield precise calculations of some physical quantities to the D4/D6 model in the next section, since technical details are given in the parallel way.

6.1 D4-brane background

We start by considering $N_{\rm c}$ D4-branes in Type IIA string theory. The $N_{\rm c}$ D4-branes provides 5D $SU(N_{\rm c})$ super Yang-Mills theory in the world volume with coupling constant

$$g^2_{\rm YM\,5} = (2\pi)^2 \alpha'^{\frac{1}{2}} g_{\rm s}. \tag{6.1}$$

Since the 5D coupling constant $g_{\rm YM\,5}$ is dimensionful, at high energies, we need to ask UV completion to a description in 11D M-theory [77]. However, this is not relevant in our discussion, since we will be interested in low energies below $1/g^2_{\rm YM\,5}$.

In order to discuss the gravity dual, we here consider the black 4-brane solutions given in (2.134) with $p = 4$ and renaming $(r, r_{\rm H}, L) \to (U, U_{\rm T}, \widetilde{R})$,

$$\mathrm{d}s^2 = \frac{1}{\sqrt{1+\dfrac{\widetilde{R}^3}{U^3}}}\left\{ -\left(1-\frac{U_{\rm T}^3}{U^3}\right)\mathrm{d}t^2 + \sum_{i=1}^{3}\mathrm{d}x^{i\,2} + \mathrm{d}\tau^2\right\} + \sqrt{1+\frac{\widetilde{R}^3}{U^3}}\left\{\frac{\mathrm{d}U^2}{\left(1-\dfrac{U_{\rm T}^3}{U^3}\right)} + U^2\mathrm{d}\Omega_4^2\right\}, \tag{6.2a}$$

$$\mathrm{e}^{\widetilde{\Phi}} = \left(1+\frac{\widetilde{R}^3}{U^3}\right)^{-1/4}, \qquad F_4 = \mathrm{d}C_3 = \frac{8\pi^3 N_{\rm c}\alpha'^{\frac{3}{2}}}{V_{S^4}}\epsilon_4, \tag{6.2b}$$

where $\epsilon_4 = \sqrt{g_{S^4}}\mathrm{d}\theta^1\mathrm{d}\theta^2\mathrm{d}\theta^3\mathrm{d}\theta^4$ is the invariant volume form of S^4 with the coordinates θ^i. The extra direction of the 5D world volume is labeled by τ. The identification of the charges (2.165) leads a

relation

$$\widetilde{R}^3(\widetilde{R}^3+U_{\rm T}^3)=R^6, \qquad \text{with} \qquad R^3=\pi N_{\rm c} g_{\rm s}\alpha'^{\frac{3}{2}}. \tag{6.3}$$

The Hawking temperature is given by

$$T=\frac{3}{4\pi}\sqrt{\frac{U_{\rm T}}{\widetilde{R}^3+U_{\rm T}^3}}. \tag{6.4}$$

As we did in the D3-brane background, we take the decoupling limit $\alpha'\to 0$ with keeping the field theory parameters, the gauge coupling constant (6.1), the temperature (6.4), and the energy U/α' finite. With the scaling behavior of the parameters; $g_{\rm s}\propto\alpha'^{-\frac{1}{2}}$, $R\propto\alpha'^{\frac{1}{3}}$, $\widetilde{R}\propto\alpha'^{\frac{1}{3}}$, $U_{\rm T}\propto\alpha'$, and $U\propto\alpha'$, we obtain the near horizon geometry as the decoupling limit,

$$\mathrm{d}s^2=\left(\frac{U}{R}\right)^{3/2}\Big\{-f(U)\mathrm{d}t^2+\sum_{i=1}^{3}\mathrm{d}x^{i\,2}+\mathrm{d}\tau^2\Big\}+\left(\frac{R}{U}\right)^{3/2}\Big\{\frac{\mathrm{d}U^2}{f(U)}+U^2\mathrm{d}\Omega_4^2\Big\}, \tag{6.5a}$$

$$\mathrm{e}^{\widetilde{\Phi}}=\left(\frac{U}{R}\right)^{3/4}, \qquad F_4=\mathrm{d}C_3=\frac{8\pi^3 N_{\rm c}\alpha'^{\frac{3}{2}}}{V_{S^4}}\epsilon_4, \tag{6.5b}$$

with Hawking temperature

$$T=\frac{3}{4\pi}\sqrt{\frac{U_{\rm T}}{R^3}}, \tag{6.6}$$

where $f(U)=1-(U_{\rm T}/U)^3$. It should be noted that the effective string coupling becomes finite

$$g_{\rm s}^{\rm eff}(U)=g_{\rm s}\mathrm{e}^{\tilde{\Phi}(U)}=\left(\frac{g_{\rm YM\,5}}{2\pi}\right)^{3/2}\frac{(U/\alpha')^{3/4}}{(\pi N_{\rm c})^{1/4}}. \tag{6.7}$$

We could obtain a relation in which the supergravity approximation is valid $g_{\rm s}^{\rm eff}(U)\ll 1$. In the case of $U_{\rm T}=0$ in the background (6.5a), the naked singularity appears at $U=0$ where the scalar curvature diverges. Therefore, in this gravity dual we could not discuss zero temperature gauge theory.

However, we can find another nontrivial solution with the same asymptotics with (6.5a) which is dual to the boundary field theory at zero/low temperature. This is obtained by applying the double Wick rotations i.e. $t\to i\tau$, $\tau\to it$ to the metric (6.2a). Taking the decoupling limit, its near horizon geometry is given by [48],

$$\mathrm{d}s^2=\left(\frac{U}{R}\right)^{3/2}\Big\{-\mathrm{d}t^2+\sum_{i=1}^{3}\mathrm{d}x^{i\,2}+f(U)\mathrm{d}\tau^2\Big\}+\left(\frac{R}{U}\right)^{3/2}\Big\{\frac{\mathrm{d}U^2}{f(U)}+U^2\mathrm{d}\Omega_4^2\Big\}, \quad \text{with} \quad f(U)=1-\left(\frac{U_{\rm KK}}{U}\right)^3. \tag{6.8}$$

Except for renaming $U_{\rm T}$ to $U_{\rm KK}$ which is a free parameter at this moment, the other parameters are the same as before. The dilaton $\widetilde{\Phi}(x)$ and the RR field $F_4(x)$ are identical to (6.5b).

We now compactify one spatial dimension τ to S^1 with radius $1/M_{\rm KK}$. Then, 4D Yang-Mills coupling constant $g_{\rm YM}$ may be related to the 5D coupling,

$$g_{\rm YM}^2=g_{\rm YM\,5}^2\frac{M_{\rm KK}}{2\pi}. \tag{6.9}$$

We impose the anti-periodic boundary conditions for the world volume fermions on this circle, so that the supersymmetry is explicitly broken. Not only fermions are massive in tree-level, but also scalar fields on the branes acquire masses through fermion one-loop. Therefore, in energies below $M_{\rm KK}$, we could expect to have the pure 4D $SU(N_{\rm c})$ Yang-Mills theory with the coupling constant (6.9).

There exists a singular point at $U = U_{\rm KK}$ in the metric (6.8). We could apply what we did in the derivation of the Hawking temperature (c.f. (4.14) and Fig.12). Since the period of τ is $2\pi/M_{\rm KK}$, a conical singularity at $U = U_{\rm KK}$ may be removed by imposing the relation

$$M_{\rm KK}^2 = \frac{9U_{\rm KK}}{4R^3}. \tag{6.10}$$

This makes the (U, τ) submanifold a cigar-like form with a tip at $U = U_{\rm KK}$ where the τ-circle shrinks to zero. The radial direction U smoothly terminates at $U = U_{\rm KK}$ and there exist no singularities. The bulk string theory parameters $U_{\rm KK}$, $g_{\rm s}$, and R are related to those of gauge theory $M_{\rm KK}$, $g_{\rm YM}$, and $N_{\rm c}$ through

$$U_{\rm KK} = \frac{2\lambda\alpha' M_{\rm KK}}{9}, \qquad g_{\rm s} = \frac{\lambda}{2\pi N_{\rm c}\sqrt{\alpha'}M_{\rm KK}}, \quad R^3 = \frac{\lambda\alpha'}{2M_{\rm KK}}, \quad \text{with} \quad \lambda = g_{\rm YM}^2 N_{\rm c}. \tag{6.11}$$

Let us consider the validity regime of supergravity approximation. We impose that the spacetime curvature should be much smaller than the string length scale. Since the maximum value of the curvature of the background (6.8) can be directly calculated as the order $(R^3U_{\rm KK})^{-1/2}$, the condition should be

$$(R^3U_{\rm KK})^{\frac{1}{2}} \gg \alpha', \quad \text{i.e.} \quad \lambda \gg 1. \tag{6.12}$$

As in the case of D3-brane, the gravity dual may be used for 4D gauge theory with large 't Hooft coupling. By using the effective string coupling (6.7), we could estimate the critical value of U where the dilaton is order one,

$$U_{\rm cutoff} = \frac{(\pi N_{\rm c})^{\frac{1}{3}} M_{\rm KK}\alpha'}{(2\pi)^2 g_{\rm YM}^2}. \tag{6.13}$$

We simply adapt the large $N_{\rm c}$ limit in the context of this paper.

It should be noted that the Kaluza-Klein modes on the D4-brane do not decouple within the supergravity approximation [48]. The masses of glueballs are of the same order as $M_{\rm KK}$, since $M_{\rm KK}$ is the only parameter in IR. Nevertheless, the qualitative features of the glueball spectrum agree with lattice calculations [78].

6.2 Confinement/deconfinement phase transition

Having the dimensionful parameter $U_{\rm KK}$ or $M_{\rm KK}$, some interesting phenomena could be discussed. Indeed, we could describe the confinement/deconfinement phase transition in terms of the geometries which is referred as Hawking-Page transition [79]. At finite temperature, there are two gravity backgrounds i.e. Euclidean versions of (6.5a) and (6.8) in which each of them has two circles. Comparing the free energies defined by the thermal partition function, it has been shown that there is a first order phase transition between two of them at $T = M_{\rm KK}/(2\pi)$, where these two become essentially the same due to $U_{\rm KK} = U_{\rm T}$. At lower temperatures $T < M_{\rm KK}/(2\pi)$, the Euclidean version of (6.8) is energetically favored, while at higher temperatures $T > M_{\rm KK}/(2\pi)$ the background (6.5a) is favored [48, 80]. This can be understood intuitively by comparing the two periods of circles i.e. $1/T$ for t and $2\pi/M_{\rm KK}$ for τ. The metric having the smaller circle which can shrink to zero is always chosen. The computation for the free energies of the two phases shows that their $N_{\rm c}$-dependence is $N_{\rm c}^0$ and $N_{\rm c}^2$ for the low temperature and the high temperature phases, respectively. This means that in the low (high) temperature phase, the gauge degrees of freedom are confined (deconfined) [48, 80]. In the previous D3-brane background, due to the conformal nature, we are always in the deconfined phase observed in the results (4.18) and (4.21). An interesting recent development regarding phase transition is reported in [81].

6.3 D8/$\overline{\text{D8}}$ flavor branes

We here briefly discuss the Sakai-Sugimoto model. We take D8 and $\overline{\text{D8}}$ branes as the flavor branes and put $N_{\rm f}$ pairs of them with the intersection in Table 2. $N_{\rm f}$ D8-branes and $N_{\rm f}$ $\overline{\text{D8}}$-branes are placed at the

	t	x^1	x^2	x^3	(τ)	U	θ^1	θ^2	θ^3	θ^4
D4	•	•	•	•	•					
D8/$\overline{\text{D8}}$	•	•	•	•		•	•	•	•	•

Table 2: The brane configurations: the background D4- and the probe D8/$\overline{\text{D8}}$-branes

positions $\tau = l/2$ and $\tau = -l/2$ with a separation distance l, respectively, These positions are generally functions of the world volume coordinates and determined by the equation of motion of the effective action of these branes.

The low energy effective theory is described by the lightest modes of open strings stretched between (D4, D4), (D4, D8) and (D4, $\overline{\text{D8}}$). As we discussed in the D3/D7 model, the gauge theories in the D8 and $\overline{\text{D8}}$-brane world volumes which come from open strings in (D8, D8) and ($\overline{\text{D8}}$, $\overline{\text{D8}}$) are decoupled, so that the symmetries in the gauge theory side are reduced to the global symmetries $U(N_{\rm f})_{\rm L}$ and $U(N_{\rm f})_{\rm R}$. Open strings in (D4, D4) give pure $SU(N_{\rm c})$ Yang-Mills as before. Those in (D4, D8) and (D4, $\overline{\text{D8}}$) which are in the case $n = 6$ in the (2.126a) and (2.126b) provide massless fermion in R sector. The NS sector only gives massive modes. Since D8 and $\overline{\text{D8}}$ branes, whose relative angle is π, have the opposite ways for GSO projection, those strings describe left-handed and right-handed chiral fermions which transform in the fundamental representation of both of $SU(N_{\rm c})$ color group and $U(N_{\rm f})_{\rm L}$ and $U(N_{\rm f})_{\rm R}$ flavor group, respectively. As in the configuration in Table 2, there are no common transverse directions where strings with finite length can live and give the mass, and so these fermions are massless. It should be mentioned that open strings in (D8, $\overline{\text{D8}}$) could produce the tachyon field. The mass of these string modes might be estimated from (2.126b) as

$$m^2 = -\frac{1}{2\alpha'} + \Big(\frac{\text{distance}}{2\pi\alpha'}\Big)^2 . \tag{6.14}$$

If the distance between D8 and $\overline{\text{D8}}$ is large enough, the system could be stable.

In the Sakai-Sugimoto model, we can view the spontaneous breaking of nonabelian chiral symmetry with a new insight. At a large radial position which corresponds to a high energy regime, D8 and $\overline{\text{D8}}$ branes are well separated and one can see the full $U(N_{\rm f})_{\rm L} \times U(N_{\rm f})_{\rm R}$ chiral symmetry. However, in the cigar-like background (6.8), as the energies are reduced i.e. going into the bulk, D8 and $\overline{\text{D8}}$ branes have no place to end and consequently join together into the $N_{\rm f}$ continuous D8 branes at some point in IR while keeping an asymptotic separation l between D8 and $\overline{\text{D8}}$ branes fixed at large U. Therefore D8-branes are embedded in the bulk by forming "U-shape". Here only the diagonal $U(N_{\rm f})$ symmetry remains unbroken. This is the geometrical realization of the spontaneous chiral symmetry breaking.

Let us consider the finite temperature case where we can discuss the confinement/deconfinement transition. At low temperature, the relevant geometry is given by the Euclidean version of (6.8) with the thermal circle for t. As we discussed above, the chiral symmetry is broken since in this regime the D8-branes make the U-shape. At high temperature, the corresponding background is (6.5a). This geometry is not the cigar type in the (τ, U)-plane which enforces D8-branes to form U-shape. In this case there are two solutions: the U-shape as before and a "parallel". In the case of the parallel, the D8 and $\overline{\text{D8}}$ branes end on the black hole horizon separately and consequently there is a full $U(N_{\rm f})_{\rm L} \times U(N_{\rm f})_{\rm R}$ chiral symmetry. Comparing the free energies of these configurations, it can be shown that in the low temperature regime the U-shaped embedding is favored, which implies that the chiral symmetry is

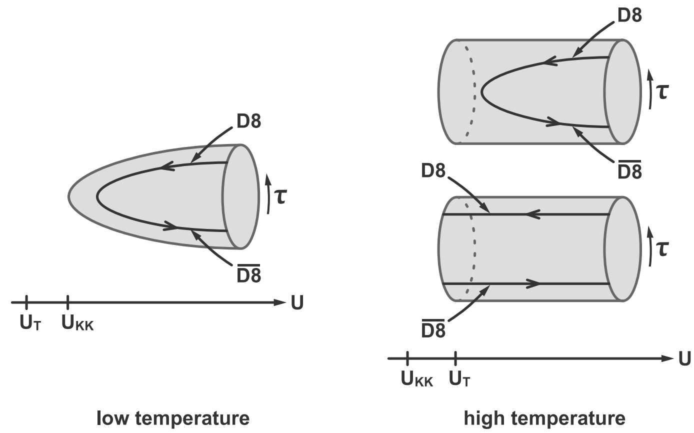

Figure 17: D8/$\overline{\text{D8}}$ configuration on confined background (left) and deconfined background (right)

broken. At high temperature, the parallel embedding is preferred, and therefore the broken symmetry is restored [80].

The baryon chemical potential and density can be also naturally introduced in the model, which endows the phase diagram with rich structures in the deconfined phase [82].

Motivated by the mass relation (6.14), it has been shown that (D8, $\overline{\text{D8}}$) bifundamental scalar mode could participate into the model in the name of "tachyon" and give the quark mass and the chiral condensate [83]. Moreover, it has been suggested that the tachyon condensate provides the mechanism of chiral symmetry breaking. Alternative approaches to introduce the quark mass into the model have been considered in [84].

6.4 Holographic baryons and nuclear force

As the applications of the Sakai-Sugimoto model in low energy QCD phenomenology, we summarize a few works on holographic baryons and two-nucleon potentials.

Holographic baryons

Roughly speaking, there are three methods to describe the property of baryons in the Sakai-Sugimoto model. We list some of results from each approach.

An immediate way is to start from a 4D effective action of mesons derived from the Sakai-Sugimoto model and do the conventional Skyrmion analysis to obtain the hedgehog soliton solution. In [85], with the 4D effective action of pions and ρ-mesons obtained from the Sakai-Sugimoto model, some interesting properties of baryons were calculated. In Table 3, we show the table II in [85]. Here M_{HH} is the ground state Skyrmion mass, and $\sqrt{\langle r^2\rangle}$ is the root-mean-square radius of the Skyrmion.

Secondly, based on the fact that the size of the instanton soliton, $\sim 1/(M_{\mathrm{KK}}\sqrt{\lambda})$, is much smaller than the typical length scale of the effective theory derived from the Sakai-Sugimoto model, $\sim 1/M_{\mathrm{KK}}$, one can build up a 5D effective action of baryons [86]. We will revisit this model soon.

Finally, one can obtain the instanton soliton in 5D which is dual to the 4D Skyrmion [87].

In Table 4, we quote a table in [88] in which static properties of neutrons and protons are extensively studied based on the 5D instanton soliton approach.

For some recent discussion on baryon properties in the Sakai-Sugimoto model, we refer to [89].

Nucleon-nucleon potentials from holography

A basic but pretty much essential problem in nuclear physics is to understand the nuclear force. So far innumerable attempts have been made to construct the nucleon-nucleon potential based on one-boson exchange picture, chiral effective theory, etc. See [90] for some reviews on the nuclear forces.

Table 3: Some results based on the 4D effective action derived from the Sakai-Sugimoto model and from a conventional Skyrmion approach as in Table II of [85].

	Results from [85]	Skyrmion approach	Experiment
f_π	92.4 MeV (input)	64.5 MeV	92.4 MeV
m_ρ	776.0 MeV (input)	-	776.0 MeV
e	7.32	5.44	-
$E_{\rm ANW} \equiv \frac{f_\pi}{2e}$	6.32 MeV	5.93 MeV	-
$r_{\rm ANW} \equiv \frac{1}{ef_\pi}$	0.29 fm	0.56 fm	-
$M_{\rm HH}$	834.0 MeV	864.3 MeV	-
$\sqrt{\langle r^2 \rangle}$	0.37 fm	0.80 fm	$0.60 \sim 0.80$ fm
M_N	-	938.9 MeV (input)	938.9 MeV
$M_\Delta - M_N$	-	293.1 MeV (input)	293.1 MeV

In this section, we briefly review the work of [91] in which the one-boson exchange model is used to calculate the nuclear force. See [92, 93] for nucleon-nucleon potentials based on the five-dimensional instanton soliton approach, where a trial soliton configuration that is supposed to describe a two-nucleon system is used to obtain the potential. We start with the four dimensional nucleon-meson action [86, 91] obtained from the Sakai-Sugimoto model

$$\int \mathrm{d}^4x \mathcal{L}_4 = \int \mathrm{d}^4x \big(- i\bar{\mathcal{N}}\gamma^\mu \partial_\mu \mathcal{N} - i m_{\mathcal{N}} \bar{\mathcal{N}}\mathcal{N} + \mathcal{L}_{\rm vector} + \mathcal{L}_{\rm axial} \big). \tag{6.15}$$

$\mathcal{L}_{\rm vector}$ and $\mathcal{L}_{\rm axial}$ denote the cubic interaction terms among vector and axial-vector mesons and baryons:

$$\mathcal{L}_{\rm vector} = -i\bar{\mathcal{N}}\gamma^\mu \beta_\mu \mathcal{N} - \sum_{k\geq 1} g_V^{(k)} \bar{\mathcal{N}}\gamma^\mu v_\mu^{(2k-1)} \mathcal{N} + \sum_{k\geq 1} g_{dV}^{(k)} \bar{\mathcal{N}}\gamma^{\mu\nu} \partial_\mu v_\nu^{(2k-1)} \mathcal{N}, \tag{6.16a}$$

$$\mathcal{L}_{\rm axial} = -\frac{ig_A}{2}\bar{\mathcal{N}}\gamma^\mu \gamma^5 \alpha_\mu \mathcal{N} - \sum_{k\geq 1} g_A^{(k)} \bar{\mathcal{N}}\gamma^\mu \gamma^5 v_\mu^{(2k)} \mathcal{N} + \sum_{k\geq 1} g_{dA}^{(k)} \bar{\mathcal{N}}\gamma^{\mu\nu}\gamma^5 \partial_\mu v_\nu^{(2k)} \mathcal{N}, \tag{6.16b}$$

where α_μ (β_μ) contain odd (even) number of pions. Before we discuss the nucleon-nucleon potential in this model, we show a few interesting predictions of the model. The tensor (or magnetic) interaction between the isospin singlet vector mesons and nucleon extracted from (6.16a) is zero, i.e. $g_{dV}^{(k)} = 0$ for isospin singlet mesons such as ω [91], which is consistent with empirical values. For instance $g_V^{(1)}/g_{dV}^{(1)} = 0.1 \pm 0.2$ for ω-mesons [94]. Another robust result is the following relation between isospin singlet coupling $g_V^{(k)}$ with $V = \omega$ and triplet one $g_V^{(k)}$ with $V = \rho$ [95],

$$\frac{g_\omega^{(k)}}{g_\rho^{(k)}} \simeq N_{\rm c} + \delta(k). \tag{6.17}$$

If one sets $N_{\rm c} = 3$ and considers the lowest mode, one finds that $g_\omega^{(1)}/g_\rho^{(1)} \simeq 3.6$; empirical values of the ratio are around 4 to 5, see [94] for example. Recently, from the effective action in (6.15) four-nucleon contact interactions are derived, and the low energy constant for each contact term is calculated [96].

From now on we focus on the result of [91] in the large $N_{\rm c}$ and large λ limit. The leading large $N_{\rm c}$ and large λ scaling of the cubic coupling is classified in [91]. For instance, for pseudo-scalars ($\varphi = \pi, \eta'$)

Table 4: Some results based on the 5D instanton soliton approach in the Sakai-Sugimoto model and from a conventional Skyrmion approach as in [88].

	instanton soliton approach	Skyrmion	experiment
$\langle r^2\rangle^{1/2}_{I=0}$	0.742 fm	0.59 fm	0.806 fm
$\langle r^2\rangle^{1/2}_{M,I=0}$	0.742 fm	0.92 fm	0.814 fm
$\langle r^2\rangle_{E,p}$	$(0.742\ \mathrm{fm})^2$	∞	$(0.875\ \mathrm{fm})^2$
$\langle r^2\rangle_{E,n}$	0	$-\infty$	$-0.116\ \mathrm{fm}^2$
$\langle r^2\rangle_{M,p}$	$(0.742\ \mathrm{fm})^2$	∞	$(0.855\ \mathrm{fm})^2$
$\langle r^2\rangle_{M,n}$	$(0.742\ \mathrm{fm})^2$	∞	$(0.873\ \mathrm{fm})^2$
$\langle r^2\rangle^{1/2}_{A}$	0.537 fm	-	0.674 fm
μ_p	2.18	1.87	2.79
μ_n	-1.34	-1.31	-1.91
$\lvert\mu_p/\mu_n\rvert$	1.63	1.43	1.46
g_A	0.734	0.61	1.27
$g_{\pi NN}$	7.46	8.9	13.2
$g_{\rho NN}$	5.80	-	$4.2 \sim 6.5$

we have

$$\begin{aligned}\frac{g_{\pi\mathcal{N}\mathcal{N}}}{2m_{\mathcal{N}}}M_{\mathrm{KK}} &= \frac{g_A^{\mathrm{triplet}}}{2f_\pi}M_{\mathrm{KK}} \simeq \frac{2\cdot 3\cdot \pi}{\sqrt{5}} \times \sqrt{\frac{N_{\mathrm{c}}}{\lambda}},\\ \frac{g_{\eta'\mathcal{N}\mathcal{N}}}{2m_{\mathcal{N}}}M_{\mathrm{KK}} &= \frac{N_{\mathrm{c}}g_A^{\mathrm{singlet}}}{2f_\pi}M_{\mathrm{KK}} \simeq \sqrt{\frac{3^9}{2}}\pi^2 \times \frac{1}{\lambda N_{\mathrm{c}}}\sqrt{\frac{N_{\mathrm{c}}}{\lambda}}.\end{aligned} \tag{6.18}$$

The leading contributions arise from the following cubic couplings

$$\frac{g_{\pi\mathcal{N}\mathcal{N}}M_{\mathrm{KK}}}{2m_{\mathcal{N}}} \sim g_{\omega^{(k)}\mathcal{N}\mathcal{N}} \sim \frac{\tilde{g}_{\rho^{(k)}\mathcal{N}\mathcal{N}}M_{\mathrm{KK}}}{2m_{\mathcal{N}}} \sim g_{a^{(k)}\mathcal{N}\mathcal{N}} \sim \sqrt{\frac{N_{\mathrm{c}}}{\lambda}}. \tag{6.19}$$

Now, we adopt the conventional one-boson exchange potential approach to obtain the holographic nucleon-nucleon potential. We take the form of the one-boson exchange potentials obtained in nuclear physics for various mesons, which is well summarized in [94], and use the meson masses and coupling constants calculated from the Sakai-Sugimoto model. For instance, the one pion exchange potential is given by

$$V_\pi^{\mathrm{holographic}} = \frac{1}{4\pi}\left(\frac{g_{\pi\mathcal{N}\mathcal{N}}M_{\mathrm{KK}}}{2m_{\mathcal{N}}}\right)^2 \frac{1}{M_{\mathrm{KK}}^2 r^3}\, S_{12}\, \vec{\tau}_1\cdot\vec{\tau}_2, \tag{6.20}$$

where $S_{12} = 3(\vec{\sigma}_1\cdot\hat{r})(\vec{\sigma}_2\cdot\hat{r}) - \vec{\sigma}_1\cdot\vec{\sigma}_2$.

The potential obtained in this way shows a similar tendency with the empirical ones. The minimum of the potential for large λ and N_{c} in [91] is at about $r \simeq 5.5/M_{\mathrm{KK}} \sim 1.1$ fm with $M_{\mathrm{KK}} \sim 1$ GeV, which is close to the minimum position of empirical two nucleon potentials. This minimum position does not change much if one considers the potential with finite values of λ and N_{c} [91]. However, the depth of the potential at the minimum position depends strongly on the value of λ and N_{c}, and it is supposed to be much shallow compared to the empirical one due to lack of scalar (or two-pion) exchange contributions. This lack of scalar attraction at the intermediate range of the nuclear force is one of the essential problem to be resolved towards a realistic nucleon-nucleon potential in holographic QCD.

7 D4/D6 model

Though the Sakai-Sugimoto model goes well with many hadronic phenomena including a spontaneous (nonabelian) chiral symmetry breaking, it is not easy to introduce quark masses into the model since no space is available between D4- and D8/$\overline{\text{D8}}$-branes. Furthermore, this model does not have attractive forces mediated by a scalar field which plays some role in low energy QCD phenomenology. Introducing different type of flavor branes gives a chance to improve these aspects: D4/D6 model [6]. Since the D4/D6 model uses the same D4 background, its dual gauge theory description is similar to that of the Sakai-Sugimoto model. On the other hand, D6 flavor branes can also have non-trivial embeddings on the D4-brane background as D3/D7 model, and so it tells us some interesting aspects of various QCD (-like) phenomena. However, the penalty we have to pay is that the D4/D6 model cannot realize nonabelian chiral symmetry like $SU(2)_{\rm L} \times SU(2)_{\rm R}$, but only $U(1)$ axial symmetry.

In D4/D6 model we use the same background (6.8) as in the Sakai-Sugimoto model,

$$\mathrm{d}s^2 = \left(\frac{U}{R}\right)^{3/2}\Big\{-\mathrm{d}t^2 + \sum_{i=1}^{3}\mathrm{d}x^{i\,2} + f(U)\mathrm{d}\tau^2\Big\} + \left(\frac{R}{U}\right)^{3/2}\left(\frac{U}{\xi}\right)^2\Big\{\mathrm{d}\xi^2 + \xi^2\mathrm{d}\Omega_4^2\Big\}, \tag{7.1}$$

where, for convenience, the dimensionless coordinate ξ has been introduced; $\mathrm{d}\xi^2/\xi^2 = \mathrm{d}U^2/f(U)U^2$ and $(U/U_{\rm KK})^{3/2} = (\xi^{3/2} + \xi^{-3/2})/2$.

7.1 D6 flavor brane

To introduce the fundamental quarks, we put $N_{\rm f}$ D6-branes on the D4-brane background [6], see Table 5 for the brane configuration. For the sake of convenience, the radial directions of the transverse part

	t	x^1	x^2	x^3	(τ)	ρ	ψ^1	ψ^2	y	ϕ
D4	•	•	•	•	•					
D6	•	•	•	•		•	•	•		

Table 5: The brane configurations: the background D4- and the probe D6-branes

are decomposed as $\xi^2 = \rho^2 + y^2$. Then, the background metric is rewritten as

$$\mathrm{d}s^2 = \left(\frac{U}{R}\right)^{3/2}\Big\{-\mathrm{d}t^2 + \sum_{i=1}^{3}\mathrm{d}x^{i\,2} + f(U)\mathrm{d}\tau^2\Big\} + \left(\frac{R}{U}\right)^{3/2}\left(\frac{U}{\xi}\right)^2\Big\{\mathrm{d}\rho^2 + \rho^2\mathrm{d}\Omega_2^2 + \mathrm{d}y^2 + y^2\mathrm{d}\phi^2\Big\}. \tag{7.2}$$

Open strings with two endpoints on (D4, D4) are considered as pure $SU(N_{\rm c})$ Yang-Mills theory as the Sakai-Sugimoto model. On the other hand, the other kind of open strings ending on (D4, D6) is in a fundamental representation with finite masses. A chiral $U(1)_{\rm A}$ symmetry is described by a rotation in (y, ϕ)-plane in this model. This chiral symmetry is broken spontaneously by non-vanishing chiral condensate which corresponds to the asymptotic slope [6], see below for more on this.

We assume that the separation y between D4- and D6-branes depends only on ρ with a fixed angle: $y = y(\rho)$ and $\phi = \phi_0$. In addition, we set $\tau = \tau_0$, that is, D6-brane is localized on the compactified circle S^1. The induced metric on D6-brane is then

$$\mathrm{d}s_{\rm D6}^2 = \left(\frac{U}{R}\right)^{3/2}\Big\{-\mathrm{d}t^2 + \sum_{i=1}^{3}\mathrm{d}x^{i\,2}\Big\} + \frac{R^{3/2}U^{1/2}}{\xi^2}\Big\{\big(1 + y'^2\big)\mathrm{d}\rho^2 + \rho^2\mathrm{d}\Omega_2^2\Big\}, \tag{7.3}$$

where the prime stands for the derivative with respect to ρ. The embedding of D6-brane is determined by $y(\rho)$. From the DBI action (2.146) for D6-brane without the bulk Kalb-Ramond field and the world volume gauge field,

$$S_{\mathrm{D6}} = -T_6 \int \mathrm{d}^7\sigma\, \mathrm{e}^{-\widetilde{\Phi}} \sqrt{-\det g_{mn}(\sigma)} = -\tau_6 \int \mathrm{d}t\, \mathrm{d}\rho\, \rho^2 \left(1 + 1/\xi^3\right)^2 \sqrt{1 + y'^2}, \quad \text{with} \quad \tau_6 = \frac{T_6 V_3 V_{S^2} U_{\mathrm{KK}}^3}{4}, \tag{7.4}$$

we can obtain the equation of motion for $y(\rho)$,

$$-6\rho^2 \left(y/\xi^5\right)\left(1 + 1/\xi^3\right)\sqrt{1 + y'^2} = \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{\rho^2 \left(1 + 1/\xi^3\right)^2 y'}{\sqrt{1 + y'^2}}\right). \tag{7.5}$$

This equation of motion can be solved numerically with given boundary conditions. For large $\rho \to \infty$, the equation of motion becomes much simpler form $0 = (\rho^2 y')'$, therefore, asymptotic behavior of the embedding is

$$y(\rho) \simeq y_\infty + \frac{c}{\rho} + \cdots . \tag{7.6}$$

Here y_∞ and c are related to the bare quark mass M_{q} and the chiral condensate $\langle\bar{\psi}\psi\rangle$, respectively.

7.2 Meson spectrum: fluctuations of D6-branes

The open string modes whose endpoints, both of them, live on N_{f} probe-branes are in the adjoint representation of $U(N_{\mathrm{f}})$ flavor symmetry. These modes can be described by the fluctuation around the classical embedding solutions of Dp-branes. Hence, we can consider meson spectra as fluctuations of the probe D6-branes

$$y(\sigma^m) = \bar{y}(\rho) + \delta y(\sigma^m), \qquad \text{and} \qquad \phi(\sigma^m) = \bar{\phi} + \delta\phi(\sigma^m), \tag{7.7}$$

where σ^m are the world volume coordinates $\sigma^m = (t, \vec{x}, \rho, \psi_1, \psi_2)$. Here $\bar{y}(\rho)$ and $\bar{\phi} = \phi_0$ denote the classical solutions of each equations of motion. Each fluctuation corresponds to scalar and pseudo-scalar meson, respectively. Then, the induced metric is written as

$$\begin{aligned} \mathrm{d}s^2_{\mathrm{D6}} = {} & \left(\frac{U}{R}\right)^{3/2}\left\{-\mathrm{d}t^2 + \sum_{i=1}^{3} \mathrm{d}x^{i\,2}\right\} + \frac{R^{3/2}U^{1/2}}{\xi^2}\left\{\left(1 + \bar{y}'^2\right)\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\Omega_2^2\right\} \\ & + \frac{R^{3/2}U^{1/2}}{\xi^2}\left\{2\bar{y}'(\partial_m \delta y)\, \mathrm{d}\rho\, \mathrm{d}\sigma^m + \left((\partial_m \delta y)(\partial_n \delta y) + (\bar{y} + \delta y)^2 (\partial_m \delta\phi)(\partial_n \delta\phi)\right)\mathrm{d}\sigma^m \mathrm{d}\sigma^n\right\}. \end{aligned} \tag{7.8}$$

Taking it into account that the bulk metric components also contain some fluctuations, we obtain a Lagrangian density of the DBI action of D6-brane up to quadratic order

$$\begin{aligned} \mathcal{L}_{\mathrm{D6}} \simeq {} & -\rho^2 \sqrt{1 + \bar{y}'^2} \\ & \times \left\{ 1 + 3\left(\frac{8\bar{y}^2 - \bar{\xi}^2}{\bar{\xi}^{10}} + \frac{4\bar{y}^2 - \rho^2}{\bar{\xi}^7}\right)\delta y^2 - \frac{\bar{y}'}{1 + \bar{y}'^2}\left(1 + \frac{1}{\bar{\xi}^3}\right)\frac{\bar{y}}{\bar{\xi}^5}\delta y \delta y' \right. \\ & + \frac{1}{2\left(1 + \bar{y}'^2\right)^2}\left(1 + \frac{1}{\bar{\xi}^3}\right)^2 \left(\frac{R^3\left(1 + \bar{y}'^2\right)}{\bar{U}\bar{\xi}^2}\partial_\mu \delta y \partial^\mu \delta y + (\delta y')^2 + \frac{1 + \bar{y}'^2}{\rho^2}\partial_i \delta y \partial^i \delta y\right) \\ & \left. + \frac{\bar{y}^2}{2\left(1 + \bar{y}'^2\right)}\left(1 + \frac{1}{\bar{\xi}^3}\right)^2 \left(\frac{R^3\left(1 + \bar{y}'^2\right)}{\bar{U}\bar{\xi}^2}\partial_\mu \delta\phi \partial^\mu \delta\phi + (\delta\phi')^2 + \frac{1 + \bar{y}'^2}{\rho^2}\partial_i \delta\phi \partial^i \delta\phi\right) \right\}, \end{aligned} \tag{7.9}$$

where μ runs through our 4D coordinates, i stands for the coordinates ψ^i, and the prime implies the derivative with respect to ρ. The linearized equations of motion are, for δy

$$\begin{aligned}
0 = & \frac{9}{2^{4/3} M_{\mathrm{KK}}^2} \frac{\rho^2}{\bar{\xi}^3} \left(1 + \frac{1}{\bar{\xi}^3}\right)^{4/3} \partial_\mu \partial^\mu \delta y + \left(1 + \frac{1}{\bar{\xi}^3}\right)^2 \partial_i \partial^i \delta y \\
& + \sqrt{1 + \bar{y}'^2} \frac{\partial}{\partial \rho} \left(\frac{\rho^2}{(1 + \bar{y}'^2)^{3/2}} \left(1 + \frac{1}{\bar{\xi}^3}\right)^2 \delta y' \right) - \sqrt{1 + \bar{y}'^2} \frac{\partial}{\partial \rho} \left(\frac{\bar{y}' \rho^2}{\sqrt{1 + \bar{y}'^2}} \left(1 + \frac{1}{\bar{\xi}^3}\right) \frac{\bar{y}}{\bar{\xi}^5} \right) \delta y \\
& - 6 \rho^2 (1 + \bar{y}'^2) \left(\frac{8 \bar{y}^2 - \bar{\xi}^2}{\bar{\xi}^{10}} + \frac{4 \bar{y}^2 - \rho^2}{\bar{\xi}^7} \right) \delta y,
\end{aligned} \tag{7.10}$$

and, for $\delta\phi$

$$0 = \frac{9}{2^{4/3} M_{\mathrm{KK}}^2} \frac{\bar{y}^2 \rho^2}{\bar{\xi}^3} \left(1 + \frac{1}{\bar{\xi}^3}\right)^{4/3} \partial_\mu \partial^\mu \delta \phi + \bar{y}^2 \left(1 + \frac{1}{\bar{\xi}^3}\right)^2 \partial_i \partial^i \delta \phi + \frac{1}{\sqrt{1 + \bar{y}'^2}} \frac{\partial}{\partial \rho} \left(\frac{\bar{y}^2 \rho^2}{\sqrt{1 + \bar{y}'^2}} \left(1 + \frac{1}{\bar{\xi}^3}\right)^2 \delta \phi' \right). \tag{7.11}$$

These linearized equations of motion can be solved numerically using the shooting method [6].

7.3 Baryon vertex: compact D4-branes

Now we introduce a baryon in our holographic description through a baryon vertex [97]. The baryon vertex in 4D corresponds to the compact D4-branes wrapping S^4 transverse to the background D4-branes (See Table 7.3 and Fig.18).

	t	x^1	x^2	x^3	(τ)	ξ	θ	θ^1	θ^2	θ^3
D4	●	●	●	●	●					
c D4	●						●	●	●	●

Table 6: The brane configurations: the background D4- and the compact D4-branes

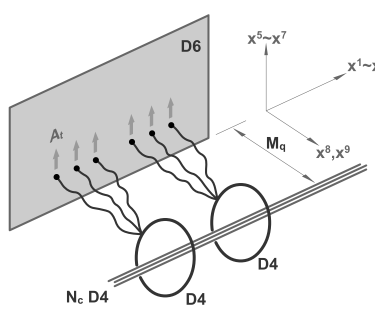

Figure 18: D4/D6 and compact D4

The induced metric on the compact D4-brane is

$$\mathrm{d}s_{\mathrm{D4}}^2 = -\left(\frac{U}{R}\right)^{3/2} \mathrm{d}t^2 + R^{3/2} U^{1/2} \left\{ \left(1 + \frac{\dot{\xi}^2}{\xi^2}\right) \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\Omega_3^2 \right\}, \tag{7.12}$$

where $\dot{\xi} = \partial\xi/\partial\theta$. In addition to the DBI action with the world volume gauge potential $A = A_t(\theta)\mathrm{d}t$, we consider the WZ term (2.152) with N_{c} fundamental strings. By using the RR flux $F_4(\sigma)$ in (6.5b),

the action of the compact D4-brane is given by [98],

$$\begin{aligned}S_{\mathrm{D4}} &= -T_4\int \mathrm{d}^5\sigma\, \mathrm{e}^{-\widetilde{\Phi}}\sqrt{-\det\left(g_{mn}+2\pi\alpha' F_{mn}\right)} + \mu_4\int(2\pi\alpha' A)\wedge F_4 \\ &= -\tau_4\int \mathrm{d}t\,\mathrm{d}\theta\, \sin^3\theta\left(\sqrt{(1+1/\xi^3)^{4/3}\left(\xi^2+\dot{\xi}^2\right)-\widetilde{F}^2}-3\widetilde{A}_t\right) \equiv \tau_4\int \mathrm{d}t\,\mathrm{d}\theta\,\mathcal{L},\end{aligned} \tag{7.13}$$

with

$$\tau_4 = \frac{1}{2^{2/3}}T_4 V_{S^3}R^3 U_{\mathrm{KK}},\quad \widetilde{A}_t = \frac{2^{2/3}(2\pi\alpha')}{U_{\mathrm{KK}}}A_t,\quad \text{and}\quad \widetilde{F}=\partial_\theta\widetilde{A}_t.$$

The dimensionless displacement is defined as

$$\frac{\partial\mathcal{L}}{\partial\widetilde{F}} = \frac{\sin^3\theta\,\widetilde{F}}{\sqrt{\left(1+1/\xi^3\right)^{4/3}(\xi^2+\dot{\xi}^2)-\widetilde{F}^2}} \equiv -D(\theta). \tag{7.14}$$

In terms of the dimensionless displacement, the equation of motion for the gauge field is expressed as

$$\partial_\theta D(\theta) = -3\sin^3\theta, \tag{7.15}$$

and the solution is given by

$$D(\theta) = D_0 + 3\left(\cos\theta - \frac{1}{3}\cos^3\theta\right). \tag{7.16}$$

The integration constant D_0 will be determined later. Now, we rewrite the Lagrangian as

$$\bar{\mathcal{L}}[\xi,\dot{\xi};\theta] = \sqrt{(\xi^2+\dot{\xi}^2)\left(1+1/\xi^3\right)^{4/3}\left(D(\theta)^2+\sin^6\theta\right)}. \tag{7.17}$$

The equation of motion from this Lagrangian determines the configuration of D4-brane which depends only on the one parameter ξ_0, the position of the compact D4 at $\theta=0$. This rewritten Lagrangian can be regarded as Hamiltonian for the baryon vertex since it is obtained by the procedure similar to the Legendre transformation. Then, the free energy of the compact D4-brane is

$$\mathcal{F}_{\mathrm{D4}} = \tau_4\int \mathrm{d}\theta\,\bar{\mathcal{L}} = \tau_4\int \mathrm{d}\theta\sqrt{(\xi^2+\dot{\xi}^2)\left(1+1/\xi^3\right)^{4/3}\left(D(\theta)^2+\sin^6\theta\right)}. \tag{7.18}$$

Considering its symmetry, the simplest situation we can imagine is that the N_{c} fundamental strings are attached only at the poles of this D4-brane, i.e. $\theta=0$ and $\theta=\pi$. At this point, the shape of the cusp is determined by the number of the strings attached. Using (7.17), we can fix the number explicitly. The tension at the cusp is obtained by the variations of the energy functional with respect to the position of the cusp as

$$f_{\mathrm{D4}} = \frac{\delta}{\delta U_c}\left[\tau_4\int \mathrm{d}\theta\,\bar{\mathcal{L}}\right]\Bigg|_{\text{fixing other values}} = \frac{N_{\mathrm{c}}T_{\mathrm{F}}\,|D_0\pm 2|}{4}\,\frac{1+1/\xi_c^3}{1-1/\xi_c^3}\,\frac{\dot{\xi}_c}{\sqrt{\xi_c^2+\dot{\xi}_c^2}}, \tag{7.19}$$

where $|D_0+2|$ and $|D_0-2|$ correspond to $\theta=0$ and $\theta=\pi$, respectively, and $\xi_c=\xi(0)$ or $\xi(\pi)$. Now we set $D_0=-2$, that is, all fundamental strings are attached at the north pole. Here T_{F} is the tension of the fundamental string. Through the numerical calculations, it is known that the force f_{D4} at the cusp is always smaller than $N_{\mathrm{c}}\times T_{\mathrm{F}}$ [102]. The baryon vertex alone cannot remain stable because the attached fundamental strings pull up the north pole to infinity [98]. One way to make it stable is to introduce probe branes which are connected to the baryon vertex through the N_{c} fundamental strings.

Since the endpoint of fundamental strings plays a role of the source for the gauge field on the probe D6-brane, we can turn on the $U(1)$ gauge field $A = A_t(\rho)\mathrm{d}t$ on the D6-brane (see Fig.18). The DBI action for this brane is

$$\begin{aligned} S_{\mathrm{D6}} &= -T_6 \int \mathrm{d}^7\sigma\, \mathrm{e}^{-\widetilde{\Phi}} \sqrt{-\det\left(g_{mn} + 2\pi\alpha' F_{mn}\right)} \\ &= -\tau_6 \int \mathrm{d}t\, \mathrm{d}\rho\, \rho^2 \left(1 + 1/\xi^3\right)^{4/3} \sqrt{\left(1 + y'^2\right)\left(1 + 1/\xi^3\right)^{4/3} - \widetilde{F}^2}\,, \end{aligned} \tag{7.20}$$

where we have defined $\widetilde{A}_t(\rho) = (2^{2/3}(2\pi\alpha')/U_{\mathrm{KK}})A_t(\rho)$ and $\widetilde{F}(\rho) = \partial_\rho \widetilde{A}_t(\rho)$. From the equation of motion for gauge fields, the conserved charge $\widetilde{Q}$ is defined as

$$\frac{\partial \mathcal{L}}{\partial \widetilde{F}} = \frac{\rho^2 \left(1 + 1/\xi^3\right)^{4/3} \widetilde{F}}{\sqrt{\left(1 + y'^2\right)\left(1 + 1/\xi^3\right)^{4/3} - \widetilde{F}^2}} \equiv \widetilde{Q}. \tag{7.21}$$

This conserved quantity $\widetilde{Q}$ is related to the number of point sources (fundamental strings) Q by

$$\widetilde{Q} = \frac{U_{\mathrm{KK}}}{2^{2/3}(2\pi\alpha')\tau_6}\, Q. \tag{7.22}$$

We can rewrite the Lagrangian with respect to $\widetilde{Q}$ with the constraint $\partial_\rho \widetilde{Q} = 0$,

$$\bar{\mathcal{L}}[y, y'; \rho; \widetilde{Q}] = \sqrt{\left(1 + y'^2\right)\left(1 + 1/\xi^3\right)^{4/3} \left(\rho^4 \left(1 + 1/\xi^3\right)^{8/3} + \widetilde{Q}^2\right)}. \tag{7.23}$$

Again, this Lagrangian plays a role of Hamiltonian. The force of D6-brane at the cusp is given by

$$f_{\mathrm{D6}} = \frac{\delta}{\delta U_c}\left[\tau_6 \int \mathrm{d}\rho\, \bar{\mathcal{L}}\right]\Big|_{\text{fixing other values}} = 2^{2/3}\tau_6 \frac{1}{U_{\mathrm{KK}}} \frac{1 + 1/\xi_c^3}{1 - 1/\xi_c^3} \frac{y_c'}{\sqrt{1 + y_c'^2}}\, \widetilde{Q} = Q T_{\mathrm{F}} \frac{1 + 1/\xi_c^3}{1 - 1/\xi_c^3} \frac{y_c'}{\sqrt{1 + y_c'^2}}. \tag{7.24}$$

Since the tension of the fundamental string is larger than that of Dp-branes, the fundamental strings shrink and two branes, on which the string was attached, meet at a single point. To be a stable configuration, the forces at the cusp should be balanced

$$\frac{Q}{N_{\mathrm{c}}} f_{\mathrm{D4}} = f_{\mathrm{D6}}(Q). \tag{7.25}$$

Once its embedding is determined, the total free energy of this D4/D6 system is obtained as a sum of each brane

$$\mathcal{F}_{\mathrm{tot}} = \frac{Q}{N_{\mathrm{c}}} \mathcal{F}_{\mathrm{D4}} + \mathcal{F}_{\mathrm{D6}}(Q). \tag{7.26}$$

Now we consider multi-flavor cases. For simplicity, we take $N_{\mathrm{f}} = 2$. This can be described by adding one more probe brane: D4/D6/D6. Since we have two probe branes, the force balancing condition is modified to

$$\frac{Q}{N_c} f_{\mathrm{D4}} = f_{\mathrm{D6}}^{(1)}(Q_1) + f_{\mathrm{D6}}^{(2)}(Q_2), \tag{7.27}$$

where $Q = Q_1 + Q_2$. With this constraint, we find minimum energy configuration. The total energy of the system is given by

$$\mathcal{F}_{\mathrm{tot}} = \frac{Q}{N_{\mathrm{c}}} \mathcal{F}_{\mathrm{D4}} + \mathcal{F}_{\mathrm{D6}}(Q_1) + \mathcal{F}_{\mathrm{D6}}(Q_2). \tag{7.28}$$

7.4 Nuclear symmetry and transition to strange matter

Now we consider a few interesting physics in the D4/D6 model: the nuclear symmetry energy [99] and a toy model study on nuclear to strange matter transition.

Symmetry energy
The nuclear symmetry energy is defined as the energy (per nucleon) difference between the isospin symmetric nuclear matter and the pure neutron matter. To describe the symmetry energy, we define an asymmetric factor $\tilde{\alpha} \equiv (N-Z)/(N+Z)$, $Z(N)$ is the number of protons (neutrons). The energy density per nucleon is written as

$$E(\rho, \tilde{\alpha}) \simeq E(\rho, 0) + E_{\rm sym}(\rho)\tilde{\alpha}^2 + \cdots , \qquad \text{where} \qquad E_{\rm sym} = \frac{1}{2}\frac{\partial^2 E}{\partial \tilde{\alpha}^2}\Big|_{\tilde{\alpha}=0} . \tag{7.29}$$

Since proton and neutron (or up and down quarks) have almost the same masses, we consider a case where two D6-branes have the same asymptotic values. In terms of $\tilde{\alpha}$, the total free energy is expressed as

$$\mathcal{F}_{\rm tot} = \frac{Q}{N_{\rm c}}\mathcal{F}_{\rm D4} + \mathcal{F}_{\rm D6}\left[\left(\frac{1+\tilde{\alpha}}{2}\right)Q\right] + \mathcal{F}_{\rm D6}\left[\left(\frac{1-\tilde{\alpha}}{2}\right)Q\right] . \tag{7.30}$$

Then, the symmetry energy follows

$$E_{\rm sym} = \frac{2\tau_6}{N_{\rm B}} \int {\rm d}\rho \, \frac{\rho^4 \sqrt{1+y'^2}\,\left(1+1/\xi^3\right)^{10/3} \widetilde{Q}^2}{\left(4\rho^4\left(1+1/\xi^3\right)^{8/3} + \widetilde{Q}^2\right)^{3/2}} , \tag{7.31}$$

where $N_{\rm B} = Q/N_{\rm c}$ is the baryon number [100]. Two main messages here are: (1) the symmetry energy keeps increasing with density, which is independent of the choice of the value of the t' Hooft coupling and $M_{\rm KK}$, (2) $E_{\rm sym} \sim \rho^{1/2}$, which is very insensitive to the value of the 't Hooft coupling and $M_{\rm KK}$. The symmetry energy was also calculated in some other holographic studies [101].

Nuclear to strange matter transition
With the D4/D6 type models, one can study a transition from nuclear to strange matter [102]. For this, two D6-branes with different asymptotic values are considered. At low density, the lower brane is attached to the baryon vertex. This means that at low density, Fermi sea consists of only light quark (up or down). As densities go up, there exist critical density at which both of the D6-branes are connected to the baryon vertex, meaning that intermediate (strange) quarks start to pile up in the Fermi sea, in addition to the light quarks.

8 Recent development: sample works

To demonstrate the versatility of the holographic QCD approach and its recent contributions to QCD (or QCD-like) phenomenology, we compile here some of recent results from various studies based on top-down or bottom-up models.

- Various hadronic form-factors were calculated in diverse approaches [103].
- The QCD trace anomaly was investigated in a holographic QCD model with the bulk dilaton field [104].
- In [105], asymptotically AdS black hole solutions of an Einstein-Maxwell-Dilaton (EMD) system was constructed and QCD (or its cousin) phase diagram at finite temperature and chemical potential was extensively studied in the newly found AdS black hole background.

- A mean field analysis for the bulk fermion field was proposed in [106]. Four-dimensional mean field of the fermion bilinear was used in an AdS/QCD model study [107].

- The authors of [108] evaluated the structure function of deep inelastic scattering from scalar mesons and polarized vector mesons in the D3/D7 model and the Sakai-Sugimoto model.

- A study on thermal QCD at large N_c was performed in [109]. In [110], some critical issues in building up a gravity dual model of QCD at finite temperature were discussed. Following the approaches described in [111,112], the author of [113] studied QCD thermodynamics comprehensively.

- An interesting attempt to build a bridge between holographic QCD and lattice Monte Carlo studies of the true QCD vacuum has been made in [114].

- Based on the light front holography approach [5,11], nucleon generalized parton distributions were investigated in [115].

- In [116], the vertex function of anomalous two vector and one axial-vector currents was calculated in the soft wall model [117] with the Chern-Simons term.

- Quark number susceptibility, a good probe for QCD phase transition, was calculated in an improved soft wall model [118].

- Detailed description on the photon-hadron high energy scattering in a gravity dual description is given in [119].

- In [120], tensor fields were incorporated into the hard wall model [8–10], and their properties were extensively studied.

- The chiral phase transition in an external magnetic field at finite temperature and chemical potential was studied in the Sakai-Sugimoto model. It is shown that for small temperatures the magnetic field lowers the critical chemical potential for chiral symmetry restoration [121] contrary to the magnetic catalysis in free space.

- In [122], thermalization of mesons with a time-dependent baryon number chemical potential was analyzed in a D3/D7 model.

- In [123], the binding energy of a holographic deuteron and tritium was calculated in the Sakai-Sugimoto model.

9 Closing Remarks

Starting from a concise summary on string theory, we tried to present a bird's eye view on holographic QCD for the non-experts, focusing more on the basic materials needed to understand the approach. We did not intend to demonstrate widely what holographic QCD can do for nuclear and hadron physics. Therefore, we showed only some out of innumerable results from holographic QCD models, see [13–17] for a review.

Despite tremendous efforts and some partial successes, it is clear that the holographic QCD approach towards realistic systems in nature is still far from its final form. There can be numerous facts that support this statement, but we will touch on only two of them. The first obvious one is that the AdS/CFT holds at large N_c and mostly stringy holographic QCD models assume the probe approximation $N_c \gg N_f$. Therefore, the holographic QCD with its present form can describe, at best, an

extreme of nonperturbative aspects of QCD. There is no guarantee that extrapolation down to $N_\mathrm{c} = 3$ is justifiable unless physical quantities at hand belong to some universality classes. However, thorough understanding of the extreme end of QCD would give some insight on nuclear-hadron physics and offer a qualitative guide to realistic QCD. For a recent attempt to overcome the large N_c issue (or go beyond the probe approximation), one may see [124]. The second issue is about the missing of essential ingredients in a holographic QCD model, which is sometimes related to the large N_c problem, to describe a realistic system. To address this issue, we give one example. It is well known that scalars with a mass of a few hundreds MeV (or two-pions) are essential to describe stable nuclear matter and to explain the intermediate-range attraction in nuclear forces. While, holographic QCD models on the market are mostly lack of this scalar attraction. Still we have no clear answer to this problem, but there have been some attempts to improve this aspect [125].

Apart from the issues mentioned above, there can be some more reasons to believe that holographic QCD in its present form may not be able to describe QCD-related realistic phenomena. However, now is certainly too early to abandon the ship. Surely to tide over fundamental issues like the probe limit, we may have to wait long for a (would-come) breakthrough in string theory. Meanwhile, holographic QCD practitioners are to explore more to equip with essential parts suitable for QCD-related realistic systems.

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A Formulae and conventions

We work with D-dimensional spacetime with signature $(-, +, \cdots, +)$. The line element is given by

$$\mathrm{d}s^2 = G_{MN}(x)\mathrm{d}x^M\mathrm{d}x^N. \tag{A.1}$$

The Christoffel symbol is defined by

$$\Gamma^M{}_{NL} = \frac{1}{2}G^{MP}\Big(\partial_N G_{PL} + \partial_L G_{PN} - \partial_P G_{NL}\Big). \tag{A.2}$$

The covariant derivative on the tensor $T^{N_1\cdots N_p}{}_{L_1\cdots L_q}(x)$ is defined by

$$\begin{aligned}\nabla_M T^{N_1\cdots N_p}{}_{L_1\cdots L_q} = \partial_M T^{N_1\cdots N_p}{}_{L_1\cdots L_q} &+ \Gamma^{N_1}{}_{MP}T^{PN_2\cdots N_p}{}_{L_1\cdots L_q} + (\text{all upper indices})\\ &- \Gamma^P{}_{ML_1}T^{N_1N_2\cdots N_p}{}_{PL_2\cdots L_q} - (\text{all lower indices}).\end{aligned} \tag{A.3}$$

The Riemann tensor is given by

$$R^P{}_{LMN} = \partial_M\Gamma^P{}_{NL} + \Gamma^P{}_{MQ}\Gamma^Q{}_{NL} - (M \longleftrightarrow N). \tag{A.4}$$

Then we can define the Ricci tensor and the scalar curvature $R_{MN} = R^P{}_{MPN}$ and $R = G^{MN}R_{MN}$, respectively.

If one performs the Weyl rescaling $G_{MN}(x) = \mathrm{e}^{\alpha\Phi(x)}\widetilde{G}_{MN}(x)$ with a constant α, various quantities transform as follows:

$$\sqrt{-G} = \mathrm{e}^{\alpha D\Phi/2}\sqrt{-\widetilde{G}},$$

$$R(G) = \mathrm{e}^{-\alpha\Phi}\Big(\widetilde{R}(\widetilde{G}) + \frac{\alpha^2}{4}(D-2)(D-1)\widetilde{G}^{MN}\partial_M\Phi\partial_N\Phi + \alpha\frac{D-1}{\sqrt{-\widetilde{G}}}\,\partial_M\big(\sqrt{-\widetilde{G}}\widetilde{G}^{MN}\partial_N\Phi\big)\Big), \quad (A.5)$$

$$\sqrt{-G}R(G) = \mathrm{e}^{\frac{\alpha(D-2)\Phi}{2}}\sqrt{-\widetilde{G}}\Big(\widetilde{R}(\widetilde{G}) - \frac{\alpha^2}{4}(D-2)(D-1)\widetilde{G}^{MN}\partial_M\Phi\partial_N\Phi\Big)$$
$$+\partial_M\Big(\alpha(D-1)\mathrm{e}^{\frac{\alpha(D-2)\Phi}{2}}\sqrt{-\widetilde{G}}\widetilde{G}^{MN}\partial_N\Phi\Big),$$

We introduce the differential form. The zero-form is just a scalar $\phi(x)$ and one-form $A(x)$ is a vector which is expanded by the basis $\mathrm{d}x^\mu$,

$$A_1(x) \equiv A_\mu(x)\mathrm{d}x^\mu. \quad (A.6)$$

Rank p antisymmetric tensor, i.e. p-form can be defined as

$$A_p(x) \equiv \frac{1}{p!}A_{\mu_1\cdots\mu_p}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}, \quad (A.7)$$

where the wedge product $\wedge$ is defend by the totally antisymmetric tensor product, for example,

$$\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \equiv \mathrm{d}x^\mu\otimes\mathrm{d}x^\nu - \mathrm{d}x^\nu\otimes\mathrm{d}x^\mu = -\mathrm{d}x^\nu\wedge\mathrm{d}x^\mu.$$

The exterior derivative $\mathrm{d} = \mathrm{d}x^\lambda\partial_\lambda$ acting on p-form is given by

$$\mathrm{d}A(x) \equiv \frac{1}{p!}\big(\partial_\lambda A_{\mu_1\cdots\mu_p}(x)\big)\mathrm{d}x^\lambda\wedge\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}. \quad (A.8)$$

It is easy to observe $\mathrm{d}^2 = 0$. The differential form is naturally defined over a manifold. If the dimensions of manifold is k, then the only nonzero integration is given by that over the k-form,

$$\int_{M_k} A_k(x) = \int_{M_k} A_{1\ldots k}(x)\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^k \equiv \int_{M_k} A_{1\ldots k}(x)\mathrm{d}^k x, \quad (A.9)$$

where it is not required to introduce the metric of the manifold. The number of the independent basis of p-form in D-dimensional spacetime is the same as that of $(D-p)$-form. One can relate these through the Hodge duality operation $*$, which plays an important role for the electric/magnetic duality:

$$*\big(\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}\big) \equiv \frac{1}{(D-p)!}\epsilon^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_{D-p}}\mathrm{d}x^{\nu_1}\wedge\cdots\wedge\mathrm{d}x^{\nu_{D-p}}, \quad (A.10)$$

where $\epsilon_{\mu_1\cdots\mu_D}(x)$ is the Levi-Civita tensor which is normalized by $\epsilon_{1\ldots D}(x) \equiv \sqrt{-G(x)}$ and $\epsilon^{1\ldots D}(x) = -1/\sqrt{-G(x)}$. The dual operation satisfies

$$**A_p = (-)^{p(D-p)+1}A_p. \quad (A.11)$$

The D-dimensional invariant volume form is given by

$$*1 = \mathrm{d}^D x\sqrt{-G}.$$

One can define the inner product of forms over M,

$$\int_M A_p\wedge *B_p = \frac{1}{p!}\int_M \mathrm{d}^D x\sqrt{-G}A_{\mu_1\cdots\mu_p}B^{\mu_1\cdots\mu_p}. \quad (A.12)$$

We also use the convention

$$|F_p|^2 \equiv \frac{1}{p!}F_{M_1\cdots M_p}F^{M_1\cdots M_p}.$$

B AdS_{d+1}

Anti-de Sitter space (AdS) appears as a vacuum solution of the Einstein equation with the negative cosmological constant. It is useful to consider d+1-dimensional AdS_{d+1} space as an embedded Lorentzian submanifold in $d+2$-dimensional flat spacetime with metric

$$\eta_{MN} = \mathrm{diag}(-,+,+,\cdots,+,-), \tag{B.1}$$

and coordinates X^M, $(M = 0,1,\ldots,d+1)$. The submanifold is then defined through the condition

$$-(X^0)^2 + (X^1)^2 + (X^3)^2 + \cdots + (X^d)^2 - (X^{d+1})^2 = -l^2, \tag{B.2}$$

where the parameter l would be the radius of AdS space. It is obvious that the defined space equips with an $SO(2,d)$ symmetry which is the isometry (the symmetry of the metric) of the AdS_{d+1} space. In addition, the space is homogeneous and isotropic. It is useful to introduce coordinates $x^m, (m = 0,1,\ldots,d)$ in AdS_{d+1} space and define an induced metric $g_{mn}(x)$ as

$$g_{mn}(x) = G_{MN}(X)\partial_m X^M \partial_n X^N, \tag{B.3}$$

where $G_{MN}(X)$ is the metric of the entire space given by (B.1) in the present case.

B.1 Global coordinates

We first consider the global coordinates (τ,ρ,x^i), $(i = 1,\ldots,d)$ covering all of the hypersurface which satisfies the condition (B.2),

$$\begin{aligned} X^0 &= l\cosh\rho\ \cos\tau, \\ X^i &= l\sinh\rho\ x^i, \\ X^{d+1} &= l\cosh\rho\ \sin\tau, \end{aligned} \tag{B.4}$$

with $0 \le \tau \le 2\pi$, $\rho \ge 0$ and $(x^1)^2 + \cdots + (x^d)^2 = 1$. The induced metric is given by

$$\mathrm{d}s^2 = l^2\Big(-\cosh^2\!\rho\ \mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2\!\rho\ \mathrm{d}\Omega_{d-1}^2\Big), \tag{B.5}$$

where we have introduced the spherical coordinates θ^i, $(i = 1,\ldots,d-1)$ on the unit S^d such as,

$$\mathrm{d}\Omega_{d-1}^2 = (\mathrm{d}\theta^1)^2 + \sin^2\theta^1(\mathrm{d}\theta^2)^2 + \sin^2\theta^1\sin^2\theta^2(\mathrm{d}\theta^3)^2 + \cdots + \sin^2\theta^1\cdots\sin^2\theta^{d-2}(\mathrm{d}\theta^{d-1})^2, \tag{B.6}$$

with $0 \le \theta^1,\cdots,\theta^{d-2} \le \pi$ and $0 \le \theta^{d-1} \le 2\pi$. It is useful to write down the volume of unit S^{d-1},

$$V_{S^{d-1}} = \int_0^{2\pi}\mathrm{d}\theta^{d-1}\int_0^{\pi}\mathrm{d}\theta^{d-2}\cdots\int_0^{\pi}\mathrm{d}\theta^1\sqrt{g_{S^{d-1}}} = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \tag{B.7}$$

where

$$\sqrt{g_{S^{d-1}}} = (\sin^{d-2}\theta^1)(\sin^{d-3}\theta^2)\cdots(\sin^2\theta^{d-3})(\sin\theta^{d-2}).$$

We display relevant volumes of S^{d-1} in the paper,

$$V_{S^3} = 2\pi^2, \qquad V_{S^4} = \frac{8\pi^2}{3}, \qquad V_{S^5} = \pi^3.$$

Since the time like circle of τ breaks the causality, we need to move on to the universal covering space to extend the circle to $-\infty < \tau < \infty$. We always refer to AdS space as this universal covering space.

We further introduce new coordinate θ as $\sinh\rho = \tan\theta$, $(0 \le \theta \le \pi/2)$. Then the metric (B.5) becomes

$$\mathrm{d}s^2 = \frac{l^2}{\cos^2\theta}\Big(-\mathrm{d}\tau^2 + \mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\Omega_{d-1}^2\Big). \tag{B.8}$$

Except for the conformal factor, the spacelike hypersurface at $\tau =$ const. in (B.8) gives the half of S^d. Its equator $(\theta = \pi/2)$ corresponds to the boundary of AdS. Along the equator, (B.8) describes the conformally compactified d-dimensional Minkowski spacetime. This is important for the $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ correspondence.

B.2 Poincaré coordinates

We next introduce the Poincaré coordinates (x^μ, u) with $\mu = 0, \underbrace{1, \cdots, d-1}_{i}$,

$$\begin{aligned} X^0 &= \frac{1}{2u}\Big(1 + u^2(L^2 + \eta_{\mu\nu}x^\mu x^\nu)\Big), \\ X^i &= Lux^i, \\ X^d &= \frac{1}{2u}\Big(1 - u^2(L^2 - \eta_{\mu\nu}x^\mu x^\nu)\Big), \\ X^{d+1} &= Lux^0. \end{aligned} \tag{B.9}$$

Then, the induced metric is given by

$$\mathrm{d}s^2 = L^2\Big(u^2\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \frac{1}{u^2}\mathrm{d}u^2\Big). \tag{B.10}$$

Further transforming the coordinates as $u \to L/z$ and $x^\mu \to x^\mu/L$, we arrive at

$$\mathrm{d}s^2 = \frac{L^2}{z^2}\Big(\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathrm{d}z^2\Big). \tag{B.11}$$

In this coordinate system, the boundary and the horizon correspond to $z = 0$ and $z = \infty$, respectively.

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